Anti-derivatives

The anti-derivative of f(x) is the function whose derivative is f(x).

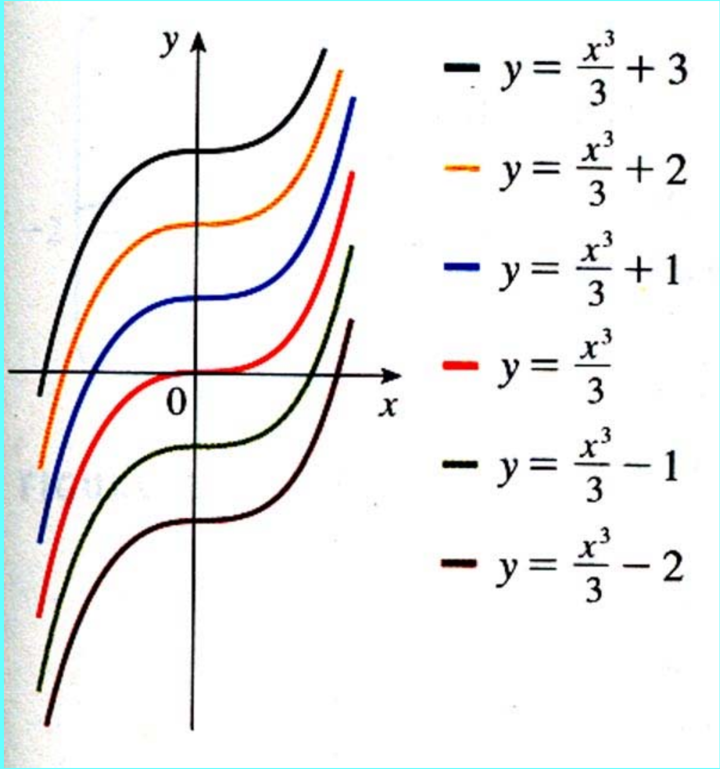
The inverse operation of differentiation, called INTEGRATION.

Ex: $f'(x) = x^2$

$$\frac{d}{dx}(x^3) = 3x^2 \rightarrow \frac{x^3}{3}$$

$$\frac{d}{dx}(x^3+1) = 3x^2 \rightarrow \frac{x^3}{3}+1$$

$$\frac{d}{dx}(x^3+2) = 3x^2 \rightarrow \frac{x^3}{3}+2$$



Any f(x) = $\frac{x^3}{3} + C$ could be the anti-derivative.

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[C] = 0$$

Integration formula

$$\int (0)\,dx = C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[kx] = k$$

Integration formula

$$\int k\,dx = kx + C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

Integration formula

$$\int \left[f(x) \pm g(x)\right] dx = \int f(x)dx + \int g(x)dx$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

Integration formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[\sin x] = \cos x$$

Integration formula

$$\int \cos x\,dx = \sin x + C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

Integration formula

$$\int \sec^2 x = \tan x + C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

Integration formula

$$\int \sec x \tan x dx = \sec x + C$$

Basic Rules of Integration

Differentiation formula

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

Integration formula

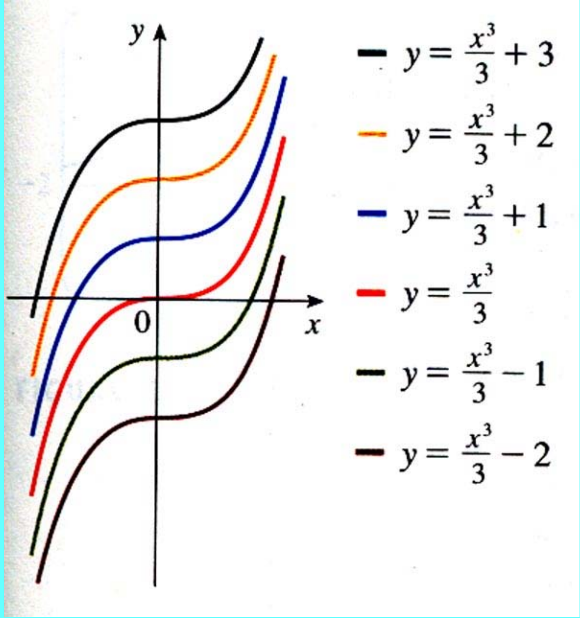
$$\int \csc^2 x = -\cot x + C$$

Ex: $$\int \frac{2}{\sqrt{x}}dx = \int 2(x)^{-\frac{1}{2}}dx = 4\sqrt{x} + C$$

Ex: $$\int (x^2+1)^2 dx = \int (x^4+2x^2+1)dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$$

Ex: $$\int \frac{x^3+3}{x^2}dx = \int \left(x+3x^{-2}\right)dx = \frac{1}{2}x^2 - \frac{3}{x} + C$$

Initial Condition Problems



Infinite solutions to any anti-derivative problem

Initial conditions are a numeric solution for a given x e.g. F(1) = 0

If $F(x) = \frac{x^3}{3} + C$ **and F(1)=0**

$$0 = \frac{1^3}{3} + C \quad \text{and} \quad C = -\frac{1}{3}$$

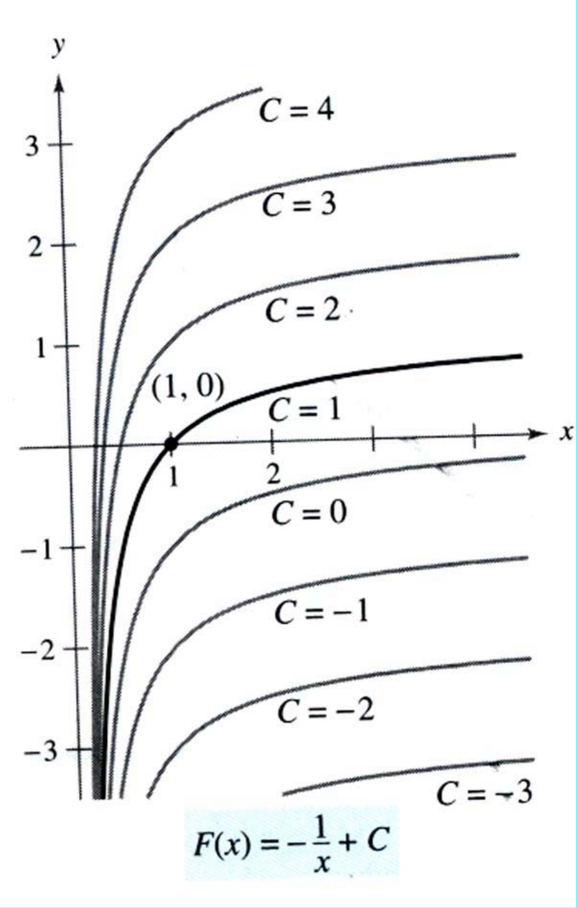
$$F(x) = \frac{x^3}{3} - \frac{1}{3}$$

Ex: Find the solution of the following if F(1)=0.

$$f'(x)=\frac{1}{x^2}$$

$$F(x)=\int \frac{1}{x^2}dx=\int x^{-2}dx=\frac{x^{-1}}{-1}=-\frac{1}{x}+C$$

$0=-\frac{1}{1}+C$ **C=1 and** $F(x)=-\frac{1}{x}+1$



Ex: A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? Acceleration due to gravity is -32 ft/sec/sec

$$a(t) = \frac{dv}{dt} = -32 \quad \longrightarrow \quad v(t) = -32t + C$$

Since v(0)=48 $\longrightarrow$ $v(t) = -32t + 48$

Since $v(t) = \frac{ds}{dt}$ $\longrightarrow$ $s(t) = -16t^2 + 48t + C$

Since s(0)=432 $\longrightarrow$ $s(t) = -16t^2 + 48t + 432$

Vertex: $v(t) = -32t + 48 = 0 \qquad t = \frac{48}{32} = 1.5$

Hits ground: $s(t) = -16t^2 + 48t + 432 = 0 \qquad t = 6.9$

4. **Show that the derivative of the right side is equal to the integrand of the left side.**

$$\int \frac{x^2 - 1}{x^{\frac{3}{2}}} dx = \frac{2\left(x^2 + 3\right)}{3\sqrt{x}} + C$$

$$f(x) = \left[\frac{2\left(x^2 + 3\right)}{3\sqrt{x}} + C \right] = \frac{2}{3}\left(x^2 + 3\right)x^{-\frac{1}{2}} + C$$

$$f'(x) = \frac{2}{3}\left[\left(x^2 + 3\right)\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + x^{-\frac{1}{2}}(2x) \right]$$

$$f'(x) = -\frac{\left(x^2 + 3\right)}{3\left(\sqrt{x}\right)^3} + \frac{2x}{\sqrt{x}} \quad = -\frac{\left(x^2 + 3\right) + 2x^2}{3\left(\sqrt{x}\right)^3} \quad = \frac{\left(x^2 - 1\right)}{\left(\sqrt{x}\right)^3}$$

8. Given: $\int x(x^2+3)dx$

Rewrite: $\int x^3\,dx + \int 3x\,dx$

Integrate: $\frac{x^4}{4} + \frac{3x^2}{2} + C$

Simplify: $\frac{x^4 + 6x^2}{4} + C$

12. Find the general solution of the differential equation and check the result by differentiation

$$\frac{dr}{d\theta} = \pi \quad \rightarrow r = \pi\theta + C$$

16. Evaluate the indefinite integral and check the result by differentiation

$$\int \left(x^2 - 2x + 3\right)dx = \frac{x^3}{3} - x^2 + 3x + C$$

$$\frac{dy}{dx}\left(\frac{x^3}{3} - x^2 + 3x + C\right) = x^2 - 2x + 3$$

Evaluate the indefinite integral and check the result by differentiation

20. $$\int\left(\sqrt[4]{x^3}+1\right)dx=\int x^{\frac{3}{4}}dx+\int dx\quad=\frac{x^{\frac{7}{4}}}{\frac{7}{4}}+x+C=\frac{4x^{\frac{7}{4}}}{7}+x+C$$

24. $$\int\frac{x^2+1}{x^2}dx\;=\int dx+\int x^{-2}dx=x+\frac{x^{-1}}{-1}=x-\frac{1}{x}+C$$

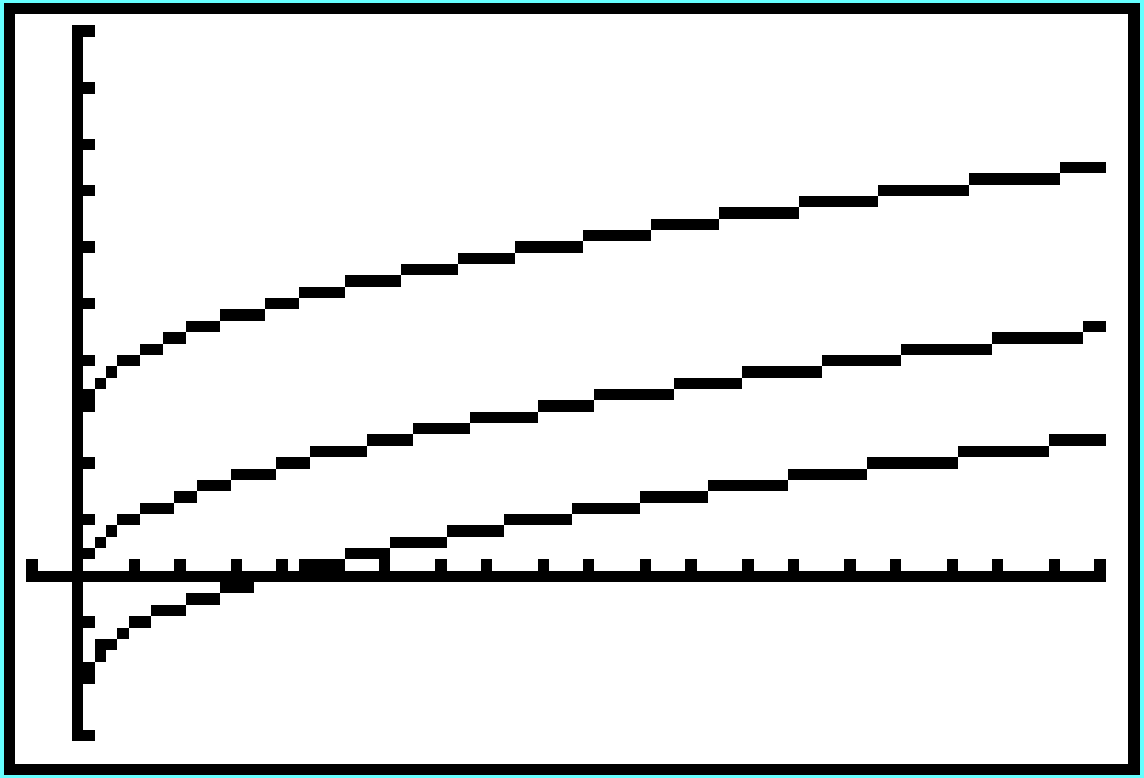
28. $$\int(1-3t)t^2dt\;=\int\left(t^2-3t^3\right)dt\;=\frac{t^3}{3}-\frac{3t^4}{4}+C$$

32. $$\int\left(t^2-\sin t\right)dt=\int t^2dt+\int(-\sin t)dt=\frac{t^3}{3}+\cos t+C$$

36. $\int \sec y(\tan y - \sec y)dy = \int (\sec y \tan y)dy - \int (\sec^2 y)dy$

$$= \sec y - \tan y + C$$

40. **Sketch the graph of the function for C = -2, 0 and 3 on the same set of axes.** $f(x) = \sqrt{x} + C$



56. **The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days (0<t<10). The initial size of the population is 500. After 1 day, the population has grown to 600. Estimate the population after 7 days.**

$$\frac{dP}{dt} = k\sqrt{t}$$

$$P = \int kt^{\frac{1}{2}}dt = k\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{2k}{3}t^{\frac{3}{2}} + C \qquad 500 = \frac{2k}{3}(0)^{\frac{3}{2}} + C \qquad C = 500$$

$$P = \frac{2k}{3}t^{\frac{3}{2}} + 500 \qquad 600 = \frac{2k}{3}(1)^{\frac{3}{2}} + 500 \qquad k = 150$$

$$P = 100t^{\frac{3}{2}} + 500$$

$$P = 100(7)^{\frac{3}{2}} + 500 = 2352$$

58. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function:

$$f(t) = -16t^2 + v_0 t + s_0$$

$$f''(t) = a(t) = -32 \text{ ft/sec}^2$$

$$f'(t) = v(t) = \frac{-32}{2} t \text{ ft/sec}^2 + C$$

Initial velocity = v_0 at t =0, $\therefore C = v_0$

$$f'(t) = v(t) = \frac{-32}{2} t \text{ ft/sec}^2 + v_0$$

$$f(t) = s(t) = -16\, t^2 \text{ ft/sec}^2 + v_0 t + C$$

Initial position at t = 0 was s_0 $\therefore C = s_0$

$$f(t) = s(t) = -16\, t^2 \text{ ft/sec}^2 + v_0 t + s_0$$

60. A balloon, rising vertically with a velocity of 16 feet per second, releases a sand bag at the instant it is 64 feet above the ground. A) How many seconds after its release will the bag strike the ground?

$$s(t) = -16t^2 + v_0 t + s_0$$

$$s(t) = -16t^2 + 16t + 64 = 0$$

$$= -16\left(t^2 - t - 4\right) = 0 \quad t = \frac{1 \pm \sqrt{17}}{2} = -1.56, 2.56$$

B) At what velocity will it hit the ground?

$$s'(t) = v(t) = -16t + v_0$$

$$v(t) = -32t + 16 = -32(2.56) + 16 = -65.97 \text{ ft/sec}$$

62. The Grand Canyon is 1600 m deep at its deepest point. A rock is dropped from the rim above this point. Express the height of the rock as a function of time in seconds. How long will it take the rock to hit the canyon floor?

Use the canyon floor as the zero height.

$$s(t) = -4.9t^2 + v_0 t + s_0$$

$$= -4.9t^2 + (0)t + 1600 = 0$$

$$t = \sqrt{\frac{1600}{4.9}} = \sqrt{326.53} = 18.1$$

64. With what initial velocity must an object be thrown upward from ground level to reach a maximum height of 200 m?

$$s(t) = -4.9t^2 + v_0 t + s_0$$

$$200 = -4.9t^2 + v_0 t$$

$$v(t) = -9.8t + v_0 = 0$$ **Velocity at the max height is zero.**

$$t = \frac{v_0}{9.8}$$

Substituting into the second equation:

$$200 = -4.9\left(\frac{v_0}{9.8}\right)^2 + v_0 \frac{v_0^2}{9.8}$$

$$-4.9v_0^2 + 9.8v_0^2 = 200(9.8)^2$$

$$v_0 = 62.61 \text{m/sec}$$

66. The minimum velocity required for an object to escape earth's gravitational pull is obtained from the solution of the equation:

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

Where v is the velocity of the object projected from the earth, y is the distance from the center of the earth, G is the gravitational constant, and M is the mass of the earth. Show that v and y are related by the equation:

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

Where v_0 is the initial velocity of the object and R is the radius of the earth.

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

Where y = R and v=v$_0$:

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

Substituting into the first equation:

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

68. Consider a particle moving along the x-axis where x(t) is the position of the particle, x'(t) is its velocity and x''(t) is its acceleration. $x(t) = (t-1)(t-3)^2 \quad 0 \le t \le 5$

A. Find the velocity and acceleration of the particle.

$$x(t) = t^3 - 7t^2 + 15t - 9$$

$$v(t) = 3t^2 - 14t + 15$$

$$a(t) = 6t - 14$$

B. Find the intervals on which the particle is moving to the right.

$$v(t) = (3t-5)(t-3)$$

Split points at 5/3 and 3. V(t) is positive for $0<t<5/3$, and for $3<t<5$.

C. Find the velocity of the particle when acceleration is zero.

$$a(t) = 6t - 14 = 0 \text{ at } t = \frac{7}{3}$$

$$v(t) = 3t^2 - 14t + 15 = 3\left(\frac{7}{3}\right)^2 - 14\left(\frac{7}{3}\right) + 15 = -\frac{4}{3}$$