Summation Notation:

$$\sum_{n=0}^{15} f(n) \qquad \sum_{i=1}^{15} f(x_i)$$

Σ, the Greek letter sigma stands for summation

Upper and Lower limits of summation

Index of summation

<u>Changing the Index of Summation</u>: We will often want to change the index of summation

Ex: Express the following so that the lower limit of summation is 0 rather than 3 and the results stay the same.

$$\sum_{k=3}^{7} 5^{k-2}$$

Define a new summation index in terms of k: j = k - 3

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j+3=3}^{7-3} 5^{(j+3)-2} = \sum_{j=0}^{4} 5^{j+1}$$

For both the sum is: $5 + 5^2 + 5^3 + 5^4 + 5^5$

Properties of Sigma Notation:

1. $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ca_3 + \dots ca_{n-1} + ca_n = c(a_1 + a_2 + \dots + a_n)$$

$$= c\sum_{k=1}^{n} a_k$$

2. $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k)$$

$$= a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n$$

$$= (a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n)$$

$$= \sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k)$$

3. $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n}(a_k) - \sum_{k=1}^{n}(b_k)$$

Proof identical to #2.

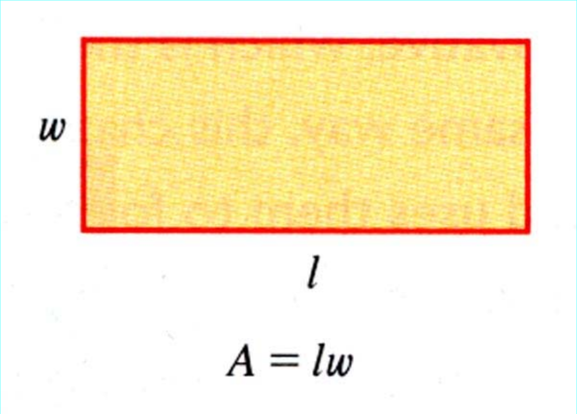
Useful Summation Formulas:

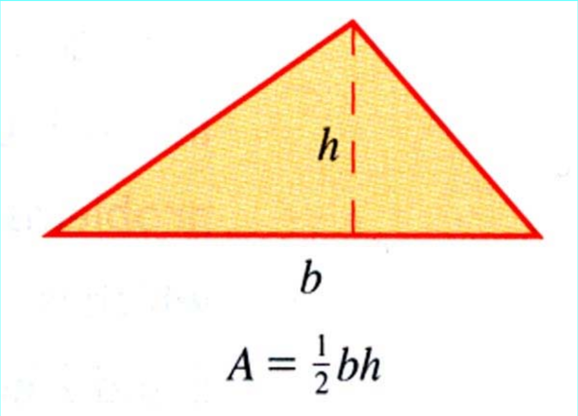
1. $$\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

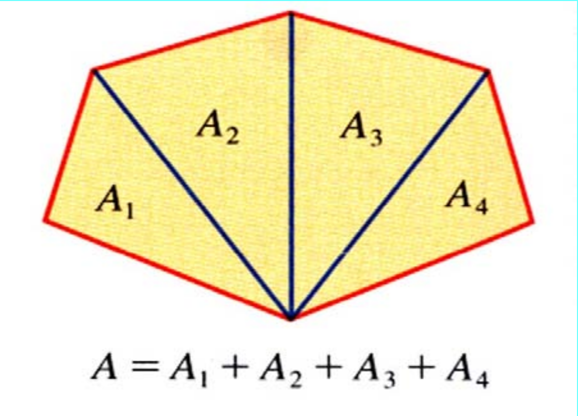
2. $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

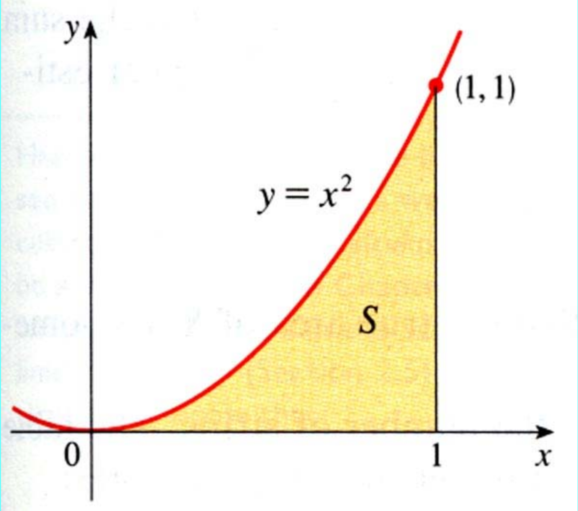
3. $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

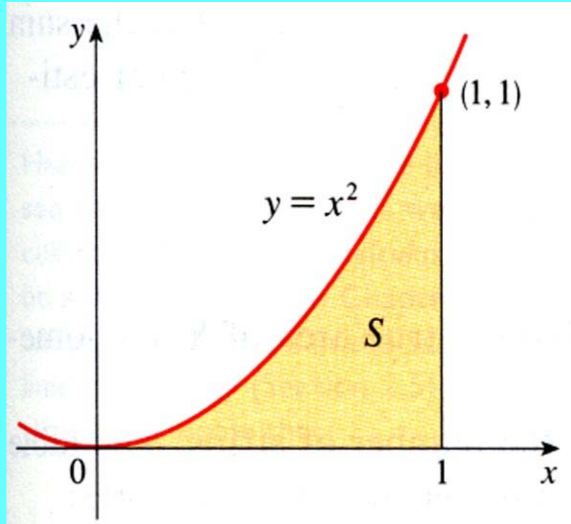
Area

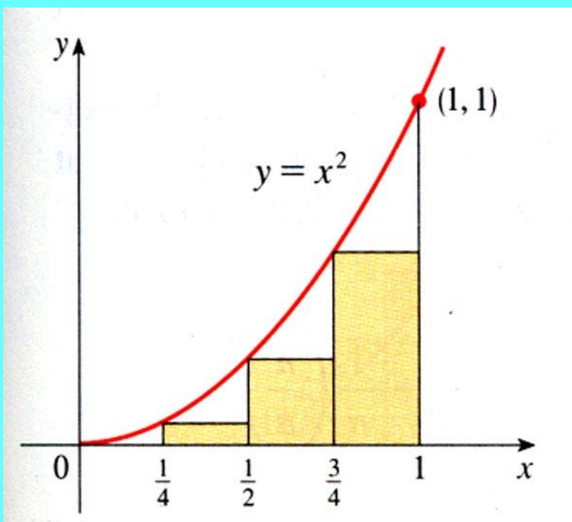




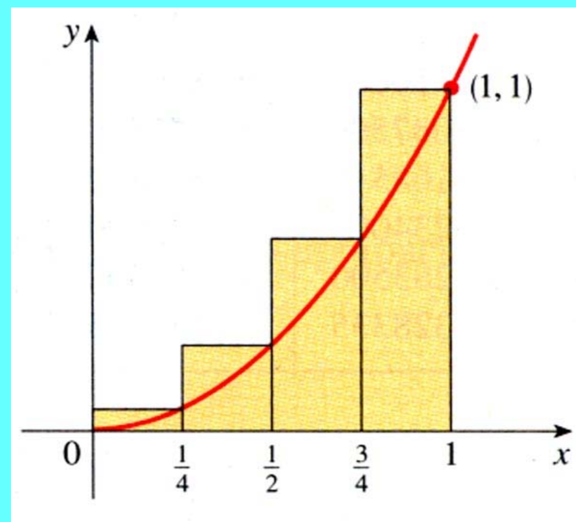






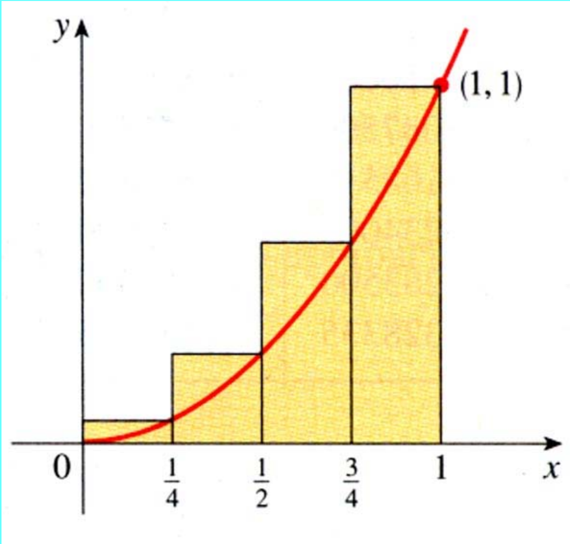


Left endpoint rectangles



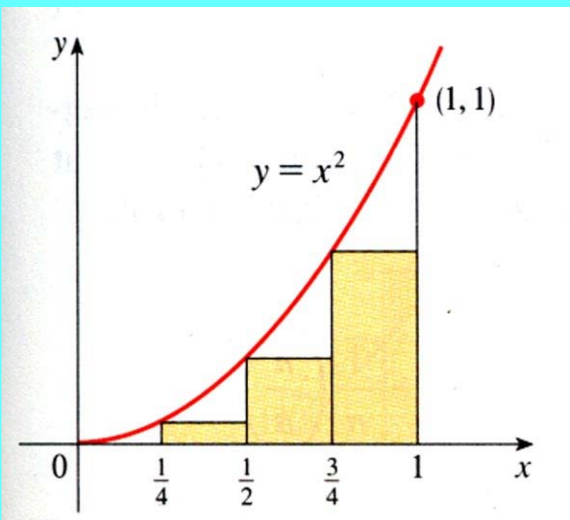
Right endpoint rectangles

N = 4 $\Delta x = 1/4$



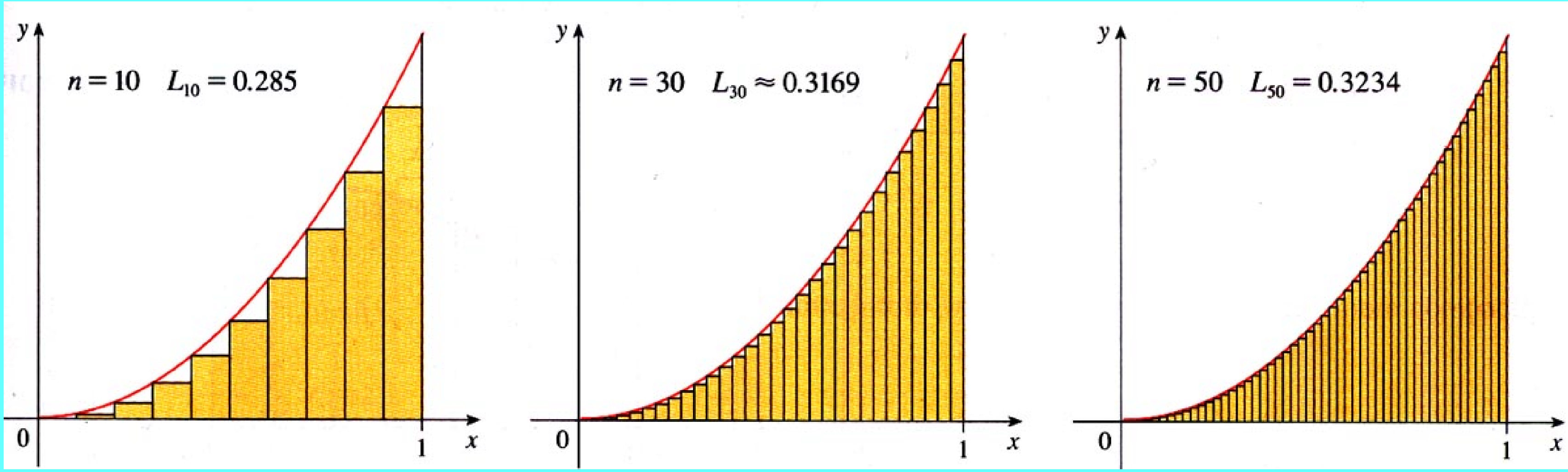
$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}1^2$$

$$= 0.46875$$

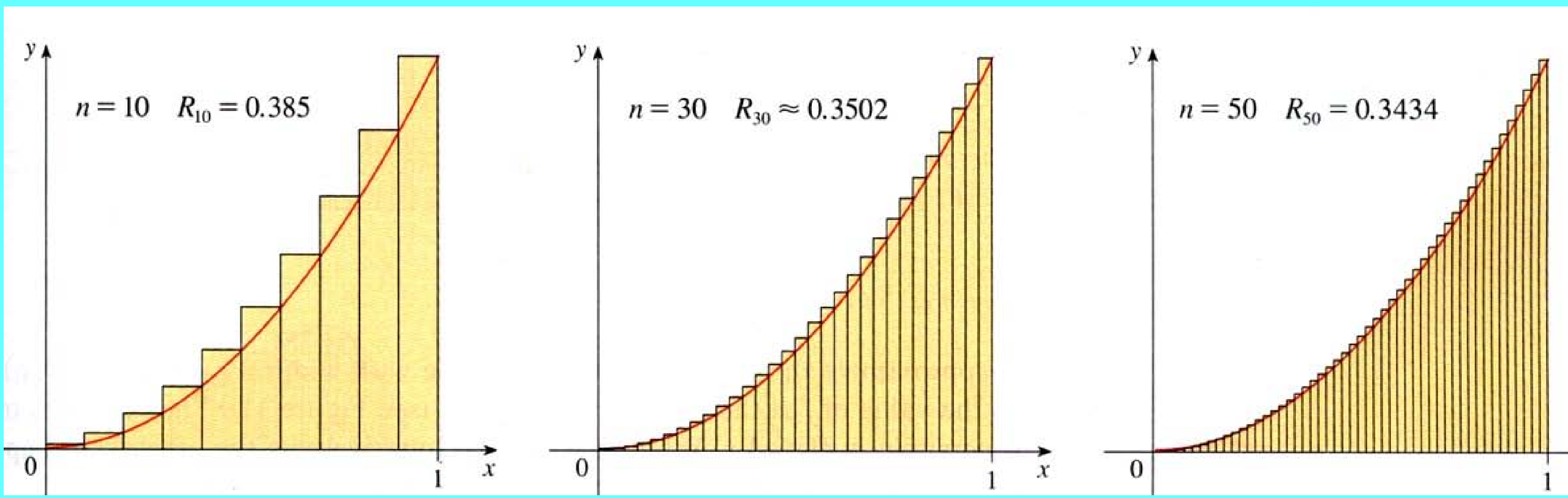


$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$

$$= 0.21875$$



$n = 10$ $L_{10} = 0.285$
$n = 30$ $L_{30} \approx 0.3169$
$n = 50$ $L_{50} = 0.3234$



$n = 10$ $R_{10} = 0.385$
$n = 30$ $R_{30} \approx 0.3502$
$n = 50$ $R_{50} = 0.3434$

Find the limit of the sum of the areas of upper approximating rectangles of y = x^2 from 0 to 1.

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right)$$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1}{6}\right) = \frac{(n+1)(2n+1}{6n^2}$$

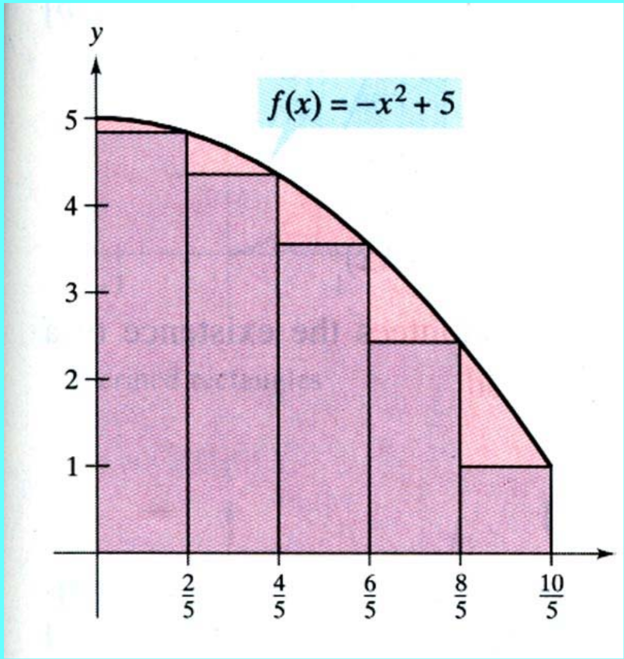
$$\underset{n\to\infty}{Limit}\left(\frac{(n+1)(2n+1}{6n^2}\right) = \underset{n\to\infty}{Limit}\,\frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

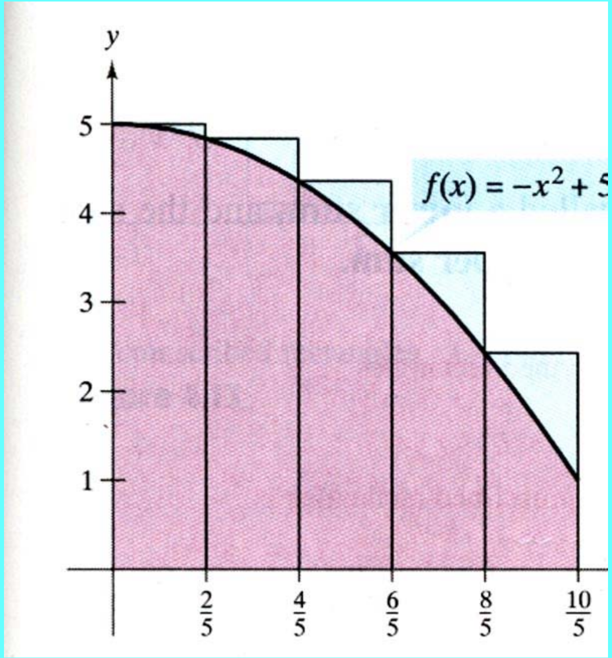
$$= \underset{n\to\infty}{Limit}\,\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$$

Excellent demonstration of various area estimation techniques

http://www.integretechpub.com/examples/interactive/AUC_java.htm

Approximating the Area of a Plane Region





Inscribed Rectangles

$$Height = f(m_i)$$

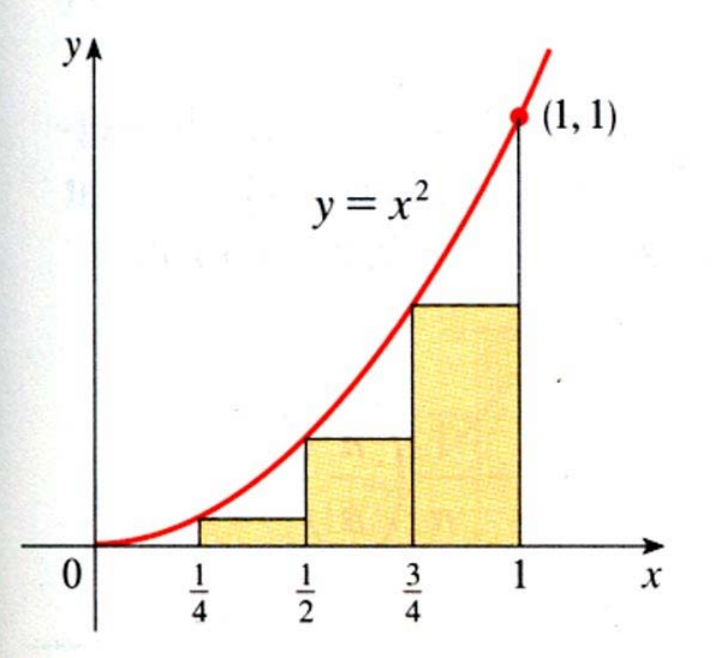
$$Area = f(m_i)\Delta x$$

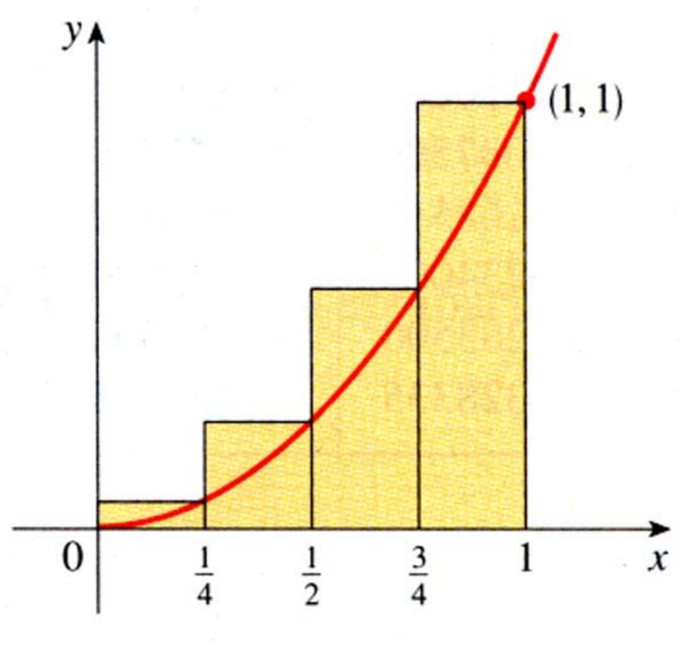
Circumscribed Rectangles

$$Height = f(M_i)$$

$$Area = f(M_i)\Delta x$$

Approximating the Area of a Plane Region





Inscribed Rectangles

$$Height = f(m_i)$$

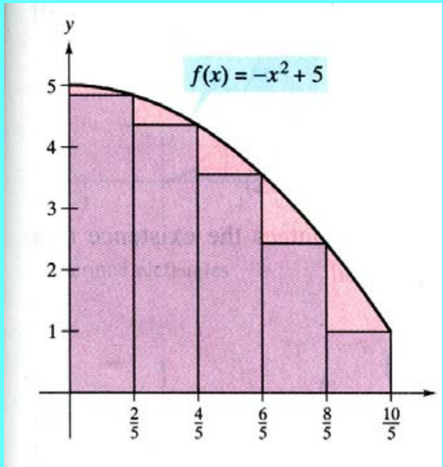
$$Area = f(m_i)\Delta x$$

Circumscribed Rectangles

$$Height = f(M_i)$$

$$Area = f(M_i)\Delta x$$

LEH Ex 3: Approx. area under f(x) = -x^2 +5 from 0 to 2 using 5 intervals.



Right sum: $m_i = a + i\left(\frac{2}{5}\right) = \frac{2i}{5}$

$$Area = \sum_{i=1}^{5} f(m_i)\Delta x$$

$$= \sum_{i=1}^{5}\left(-\left(\frac{2i}{5}\right)^2 + 5\right)\left(\frac{2}{5}\right)$$

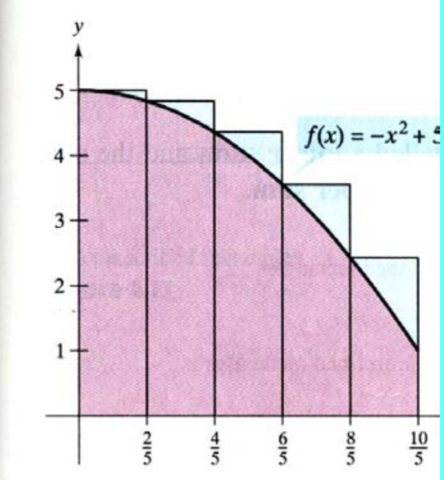
$$= \sum_{i=1}^{5}\left(-\frac{4i^2}{25}\right)\left(\frac{2}{5}\right) + \sum_{i=1}^{5} 2$$

$$= -\frac{8}{125}\sum_{i=1}^{5}\left(i^2\right) + 10$$

$$= -\frac{8}{125}\frac{n(n+1)(2n+1)}{6} + 10$$

$$= -\frac{8}{125}\frac{5(6)(11)}{6} + 10 = 6.48$$

LEH Ex 3: Approx. area under f(x) = -x^2 +5 from 0 to 2 using 5 intervals.



Left sum: $M_i = a + (i-1)\left(\frac{2}{5}\right) = \frac{2(i-1)}{5}$

$$Area = \sum_{i=1}^{5} f(M_i)\Delta x$$

$$= \sum_{i=1}^{5}\left(-\left(\frac{2(i-1)}{5}\right)^2 + 5\right)\left(\frac{2}{5}\right)$$

$$= \sum_{i=1}^{5}\left(-\frac{4(i^2-2i+1)}{25}\right)\left(\frac{2}{5}\right) + \sum_{i=1}^{5} 2$$

$$= -\frac{8}{125}\left[\sum_{i=1}^{5}\left(i^2\right) - 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1\right] + 10$$

$$= -\frac{8}{125}\left[\sum_{i=1}^{5}\left(i^2\right) - 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1\right] + 10$$

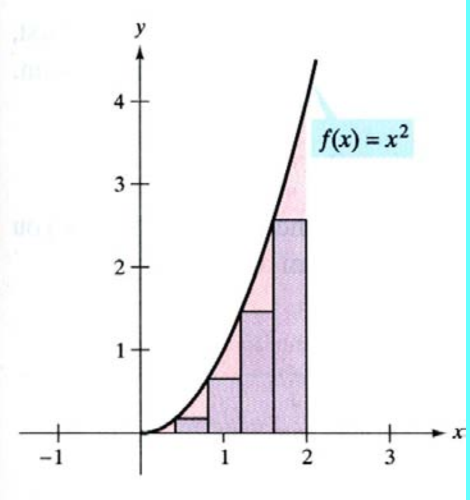
$$= -\frac{8}{125}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + 5\right] + 10$$

$$= -\frac{8}{125}\left[\frac{5(6)(11)}{6} - 2\frac{5(6)}{2} + 5\right] + 10 = 8.08$$

$$6.48 < Area < 8.08$$

Limit of Upper and Lower Sums

LHE Ex. 4: Find the limit of the upper and lower sums for the area under f(x) = x^2 from 0 to 2.



Lower sum: $m_i = a + (i-1)\left(\frac{2}{n}\right)$

$$Area = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left[\frac{2(i-1)}{n}\right]^2 \frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2 = \frac{8}{n^3}\sum_{i=1}^{n}(i^2 - 2i + 1)$$

$$= \frac{8}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right]$$

$$=\frac{8}{n^3}\left[\sum_{i=1}^{n} i^2-2\sum_{i=1}^{n} i+\sum_{i=1}^{n} 1\right]$$

$$=\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}+n\right]$$

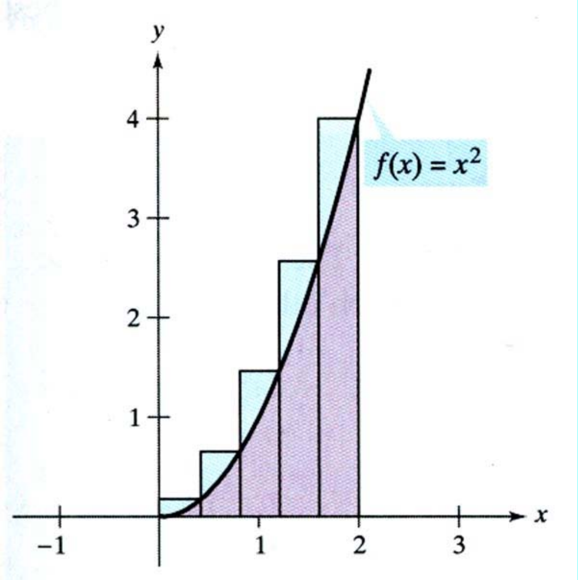
$$=\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}-6\frac{n(n+1)}{6}+\frac{6n}{6}\right]$$

$$=\frac{8}{n^3}\left[\frac{2n^3-3n^2+n}{6}\right]=\frac{4}{3}\left[2-\frac{3}{n}+\frac{1}{n^2}\right]=\frac{8}{3}$$

Limit of Upper and Lower Sums

LHE Ex. 4: Find the limit of the upper and lower sums for the area under f(x) = x^2 from 0 to 2.

Upper sum: $$m_i = a + i\left(\frac{2}{n}\right)$$



$$Area = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left[\frac{2i}{n}\right]^2 \frac{2}{n}$$

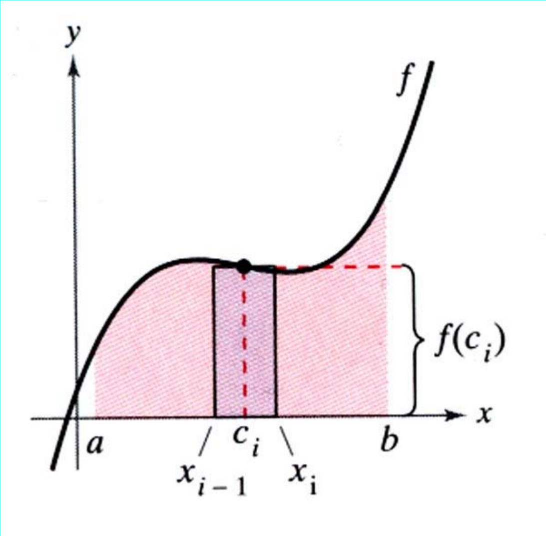
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4}{3n^3}\left[2n^3 - 3n^2 + n\right] = \frac{4}{3}\left[2 - \frac{3}{n} + \frac{1}{n^2}\right] = \frac{8}{3}$$

Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval [a,b]. The area of the region bounded by the graph of f, the x-axis and the vertical lines x=a and x=b is:



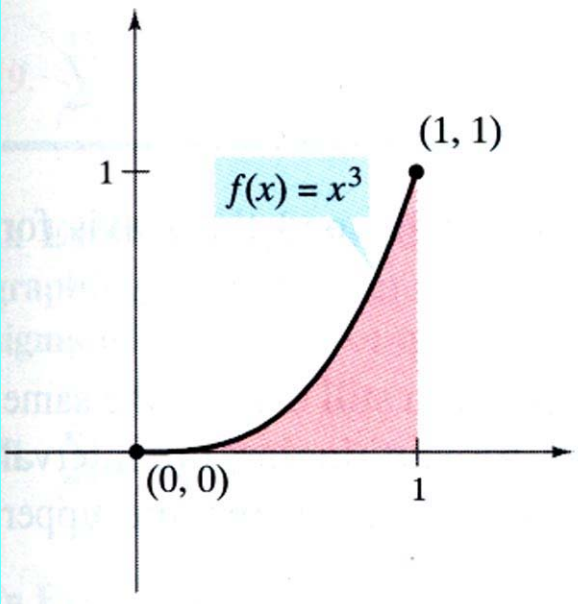
$$Area = \lim_{n\to\infty} f(c_i)\Delta x \qquad x_{i-1} < c_i < x_i$$

where $\Delta x = \dfrac{(a-b)}{n}$

Any point in the interval $x_{i-}1$ to x_i can be used to estimate the area under the curve due to the squeeze theorem

Allows you to use any convenient value in the interval .

LEH Ex 5: Find the area bounded by $f(x) = x^3$, the x-axis, x=0 and x=1.



$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

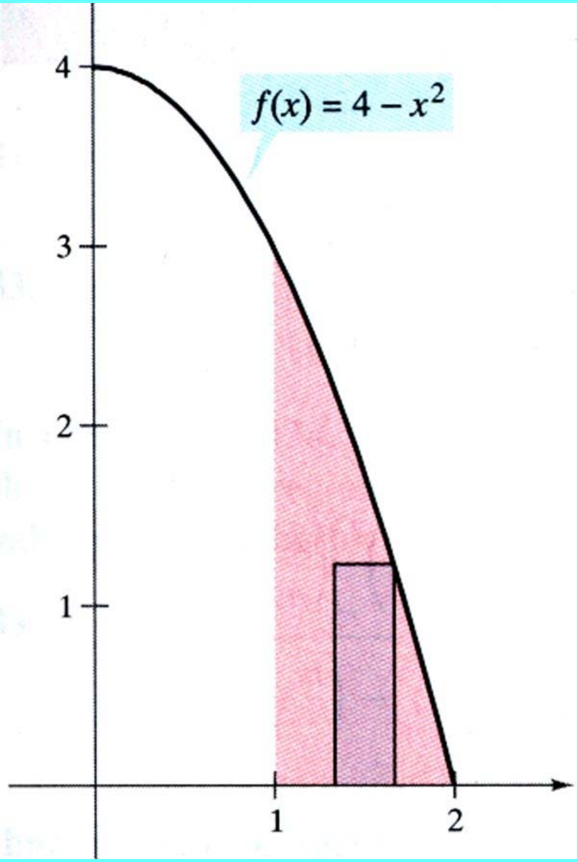
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4}\left(\frac{n^2(n^2+2n+1)}{4}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right] = \frac{1}{4}$$

LHE Ex 6: Find the area of the region bounded by $f(x)=4-x^2$, the x-axis, x=1, and x=2.



$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

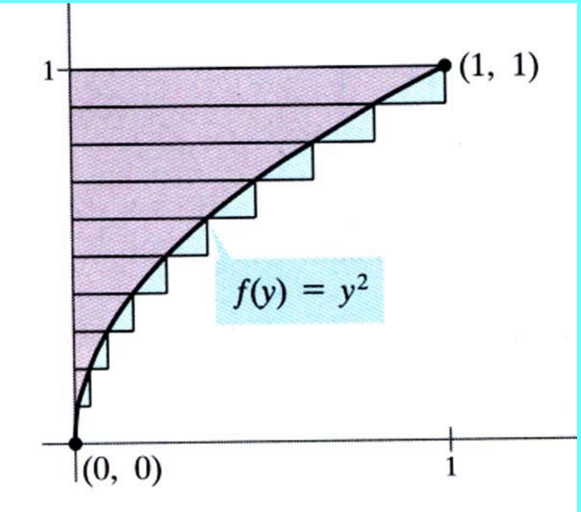
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \left(1 + \frac{i}{n}\right)^2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3 - \frac{2i}{n} + \frac{i^2}{n^2}\right]\frac{1}{n}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\sum_{i=1}^{n} 3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[3 - \left(1 + \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right] = \frac{5}{3}$$

LHE Ex 7: Find the area of the region bounded by x=y², y=1, and the y-axis.



$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^3}\right) \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$$

$$= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \lim_{n\to\infty} \left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right] = \frac{1}{3}$$

35. Find the upper and lower sums to approximate the area of the region under $y = \sqrt{x}$ from 0 to 1 with 4 divisions.

$$S(4) = \left(\frac{1}{4}\right)\sqrt{\frac{1}{4}} + \left(\frac{1}{4}\right)\sqrt{\frac{2}{4}} + \left(\frac{1}{4}\right)\sqrt{\frac{3}{4}} + \left(\frac{1}{4}\right)\sqrt{\frac{4}{4}} = 0.768$$

$$s(4) = 0\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\sqrt{\frac{1}{4}} + \left(\frac{1}{4}\right)\sqrt{\frac{2}{4}} + \left(\frac{1}{4}\right)\sqrt{\frac{3}{4}} = 0.518$$

36. Find the upper and lower sums to approximate the area of the region under $y = \sqrt{x} + 1$ from 0 to 8 with 8 divisions.

$$S(8) = \left(\frac{1}{4}\right)\left(\sqrt{\frac{1}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{2}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{3}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{4}{4}}+1\right)$$

$$+\left(\frac{1}{4}\right)\left(\sqrt{\frac{5}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{6}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{7}{4}}+1\right) + \left(\frac{1}{4}\right)\left(\sqrt{\frac{8}{4}}+1\right)$$

$$= \frac{1}{4}\left[\left(\frac{1}{2}+1\right) + \left(\frac{\sqrt{3}}{2}+1\right) + \left(\frac{\sqrt{4}}{2}+1\right) + \left(\frac{\sqrt{5}}{2}+1\right) + \left(\frac{\sqrt{6}}{2}+1\right) + \left(\frac{\sqrt{7}}{1}+1\right) + \left(\frac{\sqrt{8}}{2}+1\right)\right]$$

$$= \frac{1}{4}\left[8 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right] = 4.038$$

37. Find the upper and lower sums to approximate the area of the region under $y = \frac{1}{x}$ from 1 to 2 with 5 divisions.

$$S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{6}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{7}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{8}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{9}{5}\right)}\left(\frac{1}{5}\right)$$

$$= \left(\frac{1}{5}\right)\left[1 + \frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9}\right] \quad = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} = 0.746$$

$$s(5) = \frac{1}{\left(\frac{6}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{7}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{8}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{\left(\frac{9}{5}\right)}\left(\frac{1}{5}\right) + \frac{1}{(2)}\left(\frac{1}{5}\right)$$

$$= \frac{1}{5}\left[\frac{5}{6} + \frac{5}{7} + \frac{5}{8} + \frac{5}{9} + \frac{1}{2}\right] = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} = 0.646$$

38. Find the upper and lower sums to approximate the area of the region under $y = \sqrt{1 - x^2}$ from 0 to 1 with 5 divisions.

$$S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$$

$$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] = 0.859$$

$$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{5}{5}\right)^2}\left(\frac{1}{5}\right)$$

$$= \frac{1}{5}\left[\frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5} + 0\right] = 0.659$$

41. Use the limit process to find the area of the region between the graph of $y = -2x + 3$ and the x- axis on the interval [0, 1].

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right)+3\right]\left(\frac{1}{n}\right) = -\frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} 3$$

$$= -\frac{2}{n^2}\sum_{i=1}^{n} i + \frac{3n}{n} = 3 - \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) = 3 - \left(\frac{(n+1)}{n}\right) = 3 - 1 - \frac{1}{n} = 2 - \frac{1}{n}$$

$$\text{Limit}_{n\to\infty}\left[2 - \frac{1}{n}\right] = 2$$

42. **Use the limit process to find the area of the region between the graph of $y = 3x - 4$ and the x-axis on the interval [2, 5].**

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\sum_{i=1}^{n} i = 6 + \frac{27}{n^2}\left(\frac{n(n+1)}{2}\right) = 6 + \frac{27n^2}{2n^2} + \frac{27n}{2n^2}$$

$$\underset{n\to\infty}{\text{Limit}}\left[6 + \frac{27}{2} + \frac{27}{2n}\right] = 6 + \frac{27}{2} = \frac{39}{2}$$

43. Use the limit process to find the area of the region between the graph of $y = x^2 + 2$ and the x-axis on the interval [0, 1].

$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right) = 2 + \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= 2 + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = 2 + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\operatorname*{Limit}_{n\to\infty}\left[2 + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 2 + \frac{2}{6} = \frac{7}{3}$$

44. Use the limit process to find the area of the region between the graph of $y = 1 - x^2$ and the x-axis on the interval [-1, 1].

$$S(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(-1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 - 1 + \frac{4i}{n} - \frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\frac{8i}{n^2} - \frac{8i^2}{n^3}\right] = \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \left(\frac{4(n+1)}{n}\right) - \left(\frac{4(n+1)(2n+1)}{3n^2}\right) \qquad = \left(\frac{4(n+1)}{n}\right) - \left(\frac{4(n+1)(2n+1)}{3n^2}\right)$$

$$= 4 + \frac{4}{n} - \frac{8}{3} - \frac{8}{3n^2}$$

$$\operatorname*{Limit}_{n\to\infty}\left[4 + \frac{4}{n} - \frac{8}{3} - \frac{8}{3n^2}\right] = \frac{4}{3}$$

45. Use the limit process to find the area of the region between the graph of $y = 27 - x^3$ and the x-axis on the interval [1, 3].

$$S(n) = \sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27-\left(1+\frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \left(\frac{2}{n}\right)\sum_{i=1}^{n}\left[27-\left(1+3\frac{2i}{n}+3\frac{4i^2}{n^2}+\frac{8}{n^3}i^3\right)\right]$$

$$= 52 - \frac{12}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= 34 - \frac{26}{n} - \frac{8}{n^2}$$

$$\underset{n\to\infty}{\text{Limit}}\left[34-\frac{26}{n}-\frac{8}{n^2}\right] = 34$$

46. Use the limit process to find the area of the region between the graph of $y=2x-x^3$ and the x-axis on the interval [0, 1].

$$S(n)=\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right)-\left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) = \left(\frac{2}{n^2}\right)\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$=\left(\frac{2}{n^2}\right)\left(\frac{n(n+1)}{2}\right)-\frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) = \left(\frac{(n+1)}{n}\right)-\left(\frac{(n+1)^2}{4n^2}\right)$$

$$=1+\frac{1}{n}-\left(\frac{n^2+2n+1}{4n^2}\right) \quad = 1+\frac{1}{n}-\frac{1}{4}+\frac{1}{2n}-\frac{1}{4n^2}$$

$$\lim_{n\to\infty}\left[1+\frac{1}{n}-\frac{1}{4}+\frac{1}{2n}-\frac{1}{4n^2}\right]=\frac{3}{4}$$

47. Use the limit process to find the area of the region between the graph of $y = x^2 - x^3$ and the x-axis on the interval [-1, 1].

$$S(n) = \sum_{i=1}^{n} f\left(-1+\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1+\frac{2i}{n}\right)^2 - \left(-1+\frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \left(\frac{2}{n}\right)\sum_{i=1}^{n}\left[\left(1-\frac{4i}{n}+\frac{4i^2}{n^2}\right) - \left(-1+3(-1)^2\frac{2i}{n}+3(-1)^1\left(\frac{2i}{n}\right)^2+\left(\frac{2i}{n}\right)^3\right)\right]$$

$$= \left(\frac{2}{n}\right)\sum_{i=1}^{n}\left[1-\frac{4i}{n}+\frac{4i^2}{n^2}+1-\frac{6i}{n}+\frac{12i^2}{n^2}-\frac{8i^3}{n^3}\right]$$

$$= \left(\frac{2}{n}\right)\sum_{i=1}^{n}\left[2-\frac{10i}{n}+\frac{16i^2}{n^2}-\frac{8i^3}{n^3}\right]$$

$$= \left(\frac{2}{n}\right)\left[2\sum_{i=1}^{n}1-\frac{10}{n}\sum_{i=1}^{n}i+\frac{16}{n^2}\sum_{i=1}^{n}i^2-\frac{8}{n^3}\sum_{i=1}^{n}i^3\right]$$

$$= \left(\frac{2}{n}\right)\left[2\sum_{i=1}^{n} 1 - \frac{10}{n}\sum_{i=1}^{n} i + \frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{8}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \left(\frac{2}{n}\right)\left[2n - \frac{10}{n}\left(\frac{n(n+1)}{2}\right) + \frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{8}{n^3}\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 4 - \frac{20}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{32}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= 4 - \frac{10(n+1)}{n} + \left(\frac{16(n+1)(2n+1)}{3n^2}\right) - \left(\frac{4(n+1)^2}{n^2}\right)$$

$$= 4 - 10\left[1 + \frac{1}{n}\right] + \left(\frac{16\left(2n^2 + 3n + 1\right)}{3n^2}\right) - \left(\frac{4(n^2 + 2n + 1)}{n^2}\right)$$

$$= -6 - \frac{10}{n} + \left(\frac{32n^2 + 48n + 16}{3n^2}\right) - \left(\frac{4n^2 + 8n + 4}{n^2}\right) \quad = -10 - \frac{18}{n} + \frac{32}{3} + \frac{12}{n^2} + \frac{16}{3n^2}$$

$$\lim_{n \to \infty}\left[-10 - \frac{18}{n} + \frac{32}{3} + \frac{12}{n^2} + \frac{16}{3n^2}\right] = -10 + 10\frac{2}{3} = \frac{2}{3}$$

48. Use the limit process to find the area of the region between the graph of $y = x^2 - x^3$ and the x-axis on the interval [-1, 0].

$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1+\frac{i}{n}\right)^2 - \left(-1+\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left[\left(1-\frac{2i}{n}+\frac{i^2}{n^2}\right) - \left(-1+3(-1)^2\frac{i}{n}+3(-1)^1\left(\frac{i}{n}\right)^2+\left(\frac{i}{n}\right)^3\right)\right]$$

$$= \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left[1-\frac{5i}{n}+\frac{4i^2}{n^2}-\frac{i^3}{n^3}\right] = 2-\frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \left[2-\frac{5}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)-\frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 2-\frac{5}{2}+\frac{4}{3}-\frac{1}{4}-\frac{2}{2n}-\frac{1}{4n^2}+\frac{2}{3n^3}$$

$$\lim_{n\to\infty}\left[2-\frac{5}{2}+\frac{4}{3}-\frac{1}{4}-\frac{2}{2n}-\frac{1}{4n^2}+\frac{2}{3n^3}\right] = 2-\frac{5}{2}+\frac{4}{3}-\frac{1}{4} = \frac{7}{12}$$