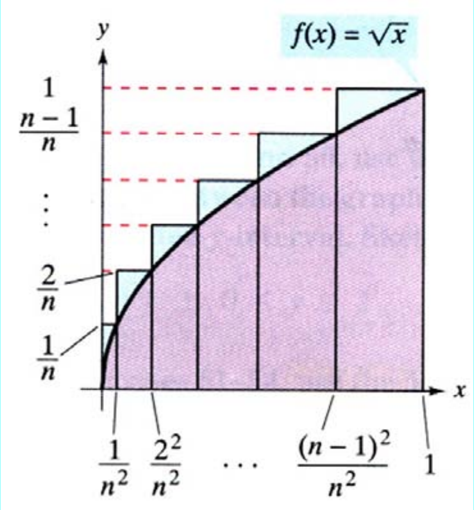
Riemann Sums

In previous work we have used equal widths for the rectangles used to estimate area. This is not necessary.

Find the area bounded by f(x), the x-axis, from 0 to 1. If $f(x) = \sqrt{x}$ **and the right endpoint is at x = i^2/n^2.**

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$



Length of Interval:

$$\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - \left(i^2 - 2i + i\right)}{n^2} = \frac{2i-1}{n^2}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\left(\frac{2i-1}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i^2-i}{n^3}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}2i^2-i \qquad = \lim_{n\to\infty}\frac{1}{n^3}\left[\sum_{i=1}^{n}2i^2-\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\left[2\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{4n^3+3n^2-n}{6}\right]$$

$$= \lim_{n\to\infty}\left[\frac{4n^3+3n^2-n}{6n^3}\right] = \lim_{n\to\infty}\left[\frac{4}{6}+\frac{1}{2n}-\frac{1}{n^3}\right] = \frac{4}{6} = \frac{2}{3}$$

Riemann Sums

If Δx_i is the width of the i^{th} subinterval on interval [a,b] and x_i^* is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

Is called the Riemann Sum for the interval.

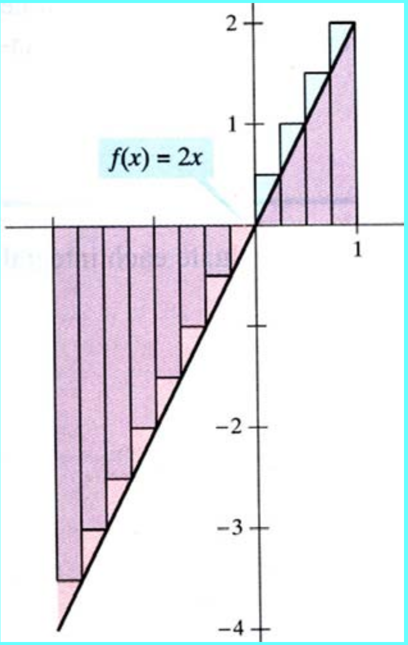
Definite Integral

$$\underset{n\to\infty}{\text{Limit}}\sum_{i=1}^{n}\Delta x\, f\left(x_i^*\right) = \int_a^b f(x)\,dx$$

The Definite Integral is a <u>number</u>.

The Indefinite Integral is a family of curves.

Ex: Evaluate: $\int_{-2}^{1} 2x\,dx$



$$\Delta x_i = \frac{b-a}{n} = \frac{3}{n} \qquad c_i = a + i\Delta x = -2 + \frac{3i}{n}$$

$$\int_{-2}^{1} 2x\,dx = \underset{n\to\infty}{Limit}\left[2\left(-2+\frac{3i}{n}\right)\frac{3}{n}\right]$$

$$= \underset{n\to\infty}{Limit}\left[-\frac{12}{n}+\frac{18i}{n^2}\right] = \underset{n\to\infty}{Limit}\left[-\frac{12}{n}\right] + \underset{n\to\infty}{Limit}\left[\frac{18i}{n^2}\right]$$

$$= \underset{n\to\infty}{Limit}\left[-\frac{12n}{n}\right] + \underset{n\to\infty}{Limit}\frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= -12 + \underset{n\to\infty}{Limit}\left[9+\frac{9}{n}\right] = -12 + 9 = -3$$

Area?

Definite Integral as Area

If f(x) is continuous and nonnegative on the closed interval [a,b], then the region bounded by the graph of f, the x-axis, and the vertical lines x=a and x=b is:

$$\int_a^b f(x)dx$$

1. **Gaps give infinite area.**
2. **Negative ordinate results in a negative product.**

Properties of Definite Integrals

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a < c < b$$

$$\int_a^b k\,f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

If f is integrable and nonnegative on [a,b]: $\int_a^b f(x)dx \geq 0$

If f(x)<g(x) for every x on [a,b], $\int_a^b f(x)dx < \int_a^b g(x)dx$

21. Given the following definite integrals find the indicated integrals.

$$\int_0^5 f(x)dx = 10 \qquad \int_5^7 f(x)dx = 3$$

a. $\int_0^7 f(x)dx$ = 13

b. $\int_5^0 f(x)dx$ = -10

c. $\int_5^5 f(x)dx$ = 0

d. $\int_0^5 3f(x)dx$ = 30

22. **Given the following definite integrals find the indicated integrals.**

$$\int_0^3 f(x)dx = 4 \qquad \int_3^6 f(x)dx = -1$$

a. $\int_0^6 f(x)dx = 3$

b. $\int_6^3 f(x)dx = 1$

c. $\int_3^3 f(x)dx = 0$

d. $\int_3^6 -5 f(x)dx = 5$

23. Given the following definite integrals find the indicated integrals.

$$\int_2^6 f(x)dx = 10 \qquad \int_2^6 g(x)dx = -2$$

a. $\int_2^6 \left[f(x) + g(x)\right]dx = 8$

b. $\int_2^6 \left[f(x) - g(x)\right]dx = 12$

c. $\int_2^6 2g(x)dx = -4$

d. $\int_2^6 3f(x)dx = 30$

24. Given the following definite integrals find the indicated integrals.

$$\int_{-1}^{1} f(x)dx = 0 \qquad \int_{0}^{1} f(x)dx = 5$$

a. $\int_{-1}^{0} f(x)dx$ = -5

b. $\int_{0}^{1} f(x)dx - \int_{-1}^{0} f(x)dx$ = 10

c. $\int_{-1}^{1} 3g(x)dx$ = 0

d. $\int_{0}^{1} 3f(x)dx$ = 15

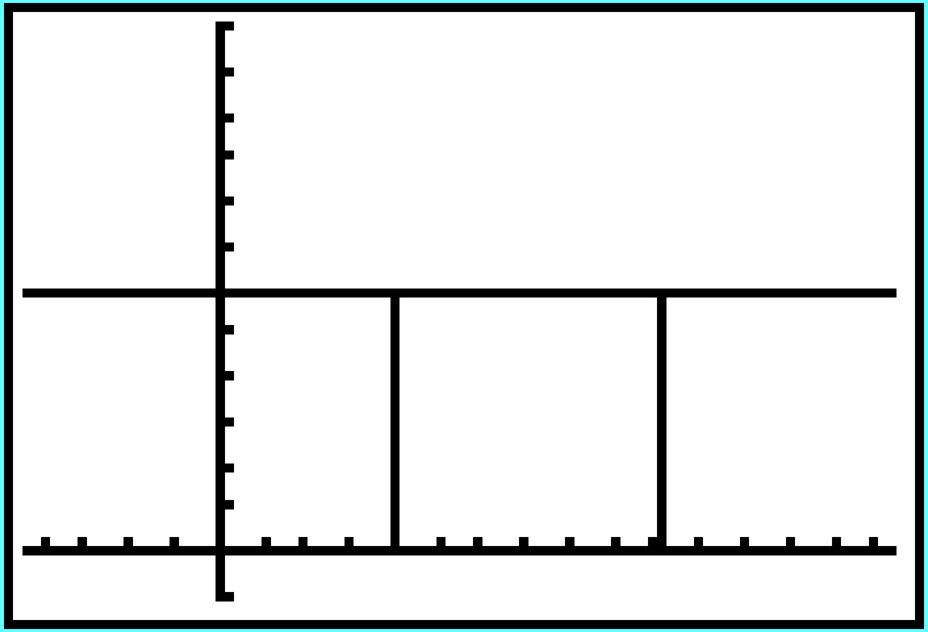
25. Evaluate by the limit definition. $\displaystyle\int_4^{10} 6\,dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right)$$

$$= \sum_{i=1}^{n} 6\left(\frac{6}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{36}{n}$$

$$= 36$$



26. Evaluate by the limit definition. $\int_{-2}^{3} x\,dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} f\left(-2+\frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-\frac{10}{n}+\frac{25i}{n^2}\right)$$

$$= -10+\frac{25}{n^2}\sum_{i=1}^{n} i \qquad = -10+\frac{25}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= -10+\left(\frac{25(n+1)}{2n}\right) \qquad = -10+\frac{25}{2}+\frac{25}{2n} = \frac{5}{2}+\frac{25}{2n}$$

$$\lim_{n\to\infty}\left(\frac{5}{2}+\frac{25}{2n}\right) = \frac{5}{2}$$

27. Evaluate by the limit definition. $\int_{-1}^{1} x^3 dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$

$$=\left(\frac{2}{n}\right)\sum_{i=1}^{n} -1+3\frac{2i}{n}-3\left(\frac{2i}{n}\right)^2+\left(\frac{2i}{n}\right)^3 \quad = \sum_{i=1}^{n} -\frac{2}{n}+\frac{12i}{n^2}-\frac{24i^2}{n^3}+\frac{16i^3}{n^4}$$

$$= -2+\frac{12}{n^2}\sum_{i=1}^{n} i-\frac{24}{n^3}\sum_{i=1}^{n} i^2+\frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2+\frac{12}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= -2+6\left(1+\frac{1}{n}\right)-4\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+4\left(1+\frac{2}{n}+\frac{1}{n^2}\right)=\frac{2}{n}$$

$$\lim_{n\to\infty}\left(\frac{2}{n}\right)=0$$

28. Evaluate by the limit definition. $\int_0^1 x^3 dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{i^3}{n^4}$$

$$= \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \frac{1}{n^4} \left(\frac{n^2 (n+1)^2}{4} \right)$$

$$= \frac{1}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right)$$

$$\text{Limit}_{n\to\infty} \left(\frac{1}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \right) = \frac{1}{4}$$

29. Evaluate by the limit definition. $\int_1^2 (x^2+1)\,dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(1+\frac{i}{n}\right)^2+1\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[1+\frac{2i}{n}+\frac{i^2}{n^2}+1\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\frac{2}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right]$$

$$= 2+\frac{2}{n^2}\sum_{i=1}^{n} i+\frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= 2+\frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2+1+\frac{1}{n}+\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$

$$\operatorname*{Limit}_{n\to\infty}\left(\frac{10}{3}+\frac{1}{n}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{10}{3}$$

30. Evaluate by the limit definition. $\int_1^2 4x^2dx$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n} 4\left(1+\frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} 4\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \qquad = \sum_{i=1}^{n}\left(\frac{4}{n}+\frac{8i}{n^2}+\frac{4i^2}{n^3}\right)$$

$$= 4+\frac{8}{n^2}\sum_{i=1}^{n} i+\frac{4}{n^3}\sum_{i=1}^{n} i^2$$

$$= 4+\frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 4+4+\frac{1}{n}+\frac{4}{3}+\frac{2}{n}+\frac{2}{3n^2} = \frac{28}{3}+\frac{1}{n}+\frac{2}{n}+\frac{2}{3n^2}$$

$$\underset{n\to\infty}{\text{Limit}}\left(\frac{28}{3}+\frac{1}{n}+\frac{2}{n}+\frac{2}{3n^2}\right) = \frac{28}{3}$$

Express the limit as a definite integral on the interval [a,b].

31. $\underset{\|\Delta\|\to 0}{\text{Limit}} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i \quad [-1, 5] \quad = \int_{-1}^{5} (3x+10)\,dx$

32. $\underset{\|\Delta\|\to 0}{\text{Limit}} \sum_{i=1}^{n} 6c_i(4-c_i)^2\,\Delta x_i \quad [0, 4] \quad = \int_{0}^{4} 6x(4-x)^2\,dx$

33. $\underset{\|\Delta\|\to 0}{\text{Limit}} \sum_{i=1}^{n} \sqrt{c_i^2 - 4}\,\Delta x_i \quad [0, 3] \quad = \int_{0}^{3} \sqrt{x^2-4}\,dx$

34. $\underset{\|\Delta\|\to 0}{\text{Limit}} \sum_{i=1}^{n} \frac{3}{c_i^2}\,\Delta x_i \quad [1, 3] \quad = \int_{1}^{3} \frac{3}{x^2}\,dx$

In exercises 39 - 42 connect the figures with <, >, or =, if the interval [1, 5] has been partitioned into n subintervals of equal width, Δx, and x_i is the stated point in the subinterval.

39. Left endpoint

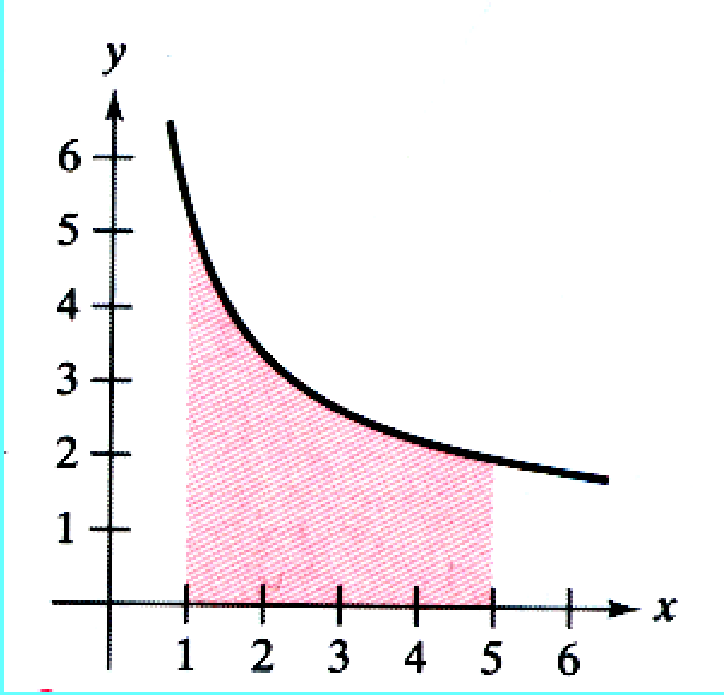
$$\sum_{i=1}^{n} f(x_i)\Delta x_i \;>\; \int_1^5 f(x)\,dx$$

40. Right endpoint

$$\sum_{i=1}^{n} f(x_i)\Delta x_i \;<\; \int_1^5 f(x)\,dx$$

41. Midpoint

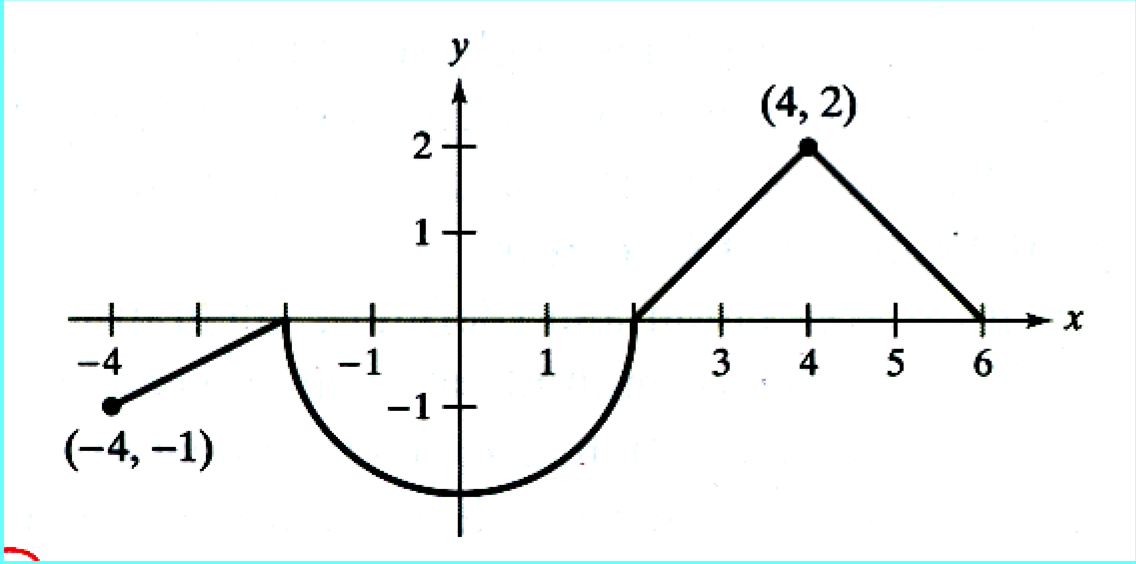
$$\sum_{i=1}^{n} f(x_i)\Delta x_i \;<\; \int_1^5 f(x)\,dx$$



42. T is the average of 39-41

$$T \;>\; \int_1^5 f(x)\,dx$$

43. Evaluate each definite integral using geometric formulas.



a. $\int_{0}^{2} f(x)dx = -\pi$

b. $\int_{2}^{6} f(x)dx = 4$

c. $\int_{-4}^{2} f(x)dx = -(1+2\pi)$

d. $\int_{-4}^{6} f(x)dx = 3-2\pi$

e. $\int_{-4}^{6} [f(x)+2]dx = 23-2\pi$

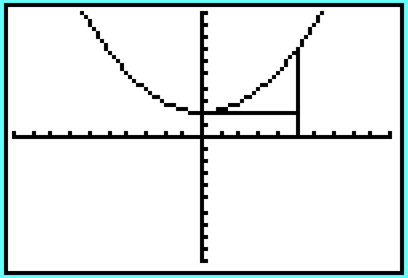
44. Consider the function f that is continuous on the interval [-5, 5] and for which the area under the curve from [0, 5] is four. Evaluate each of the following definite integrals.

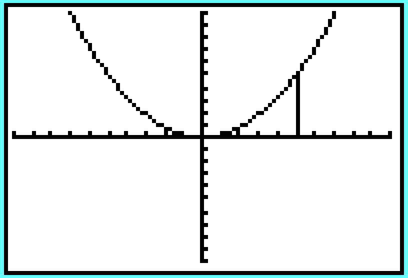
a. $\int_{0}^{5} [f(x)+2]dx = 14$

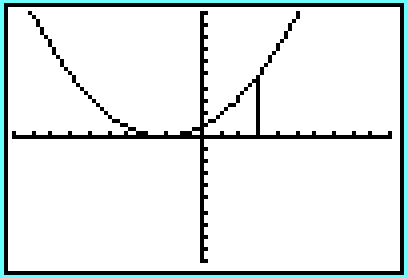
b. $\int_{-2}^{3} f(x+2)dx = 4$

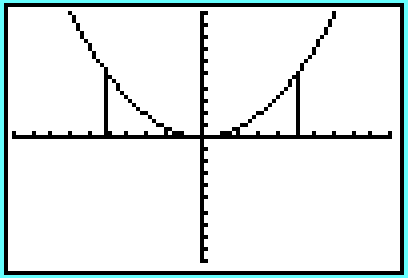
c. $\int_{-5}^{5} f(x)dx = 8$ If x is even.

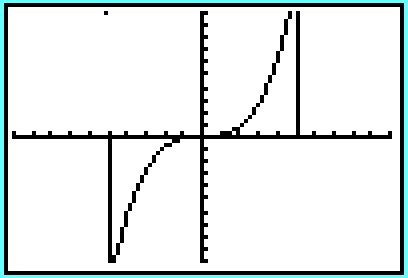
d. $\int_{-5}^{5} f(x)dx = 0$ If x is odd.





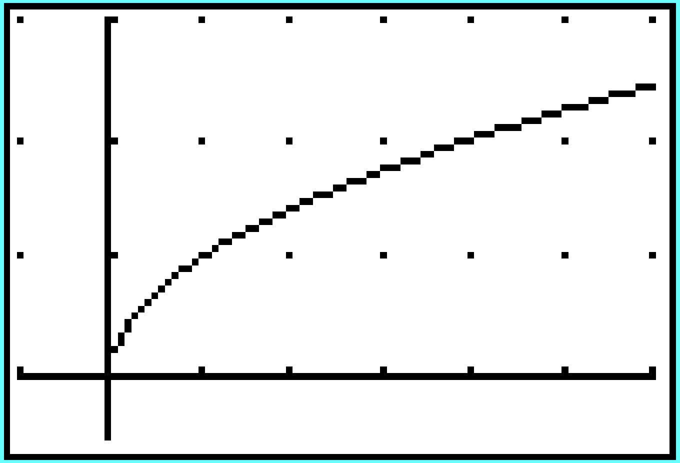




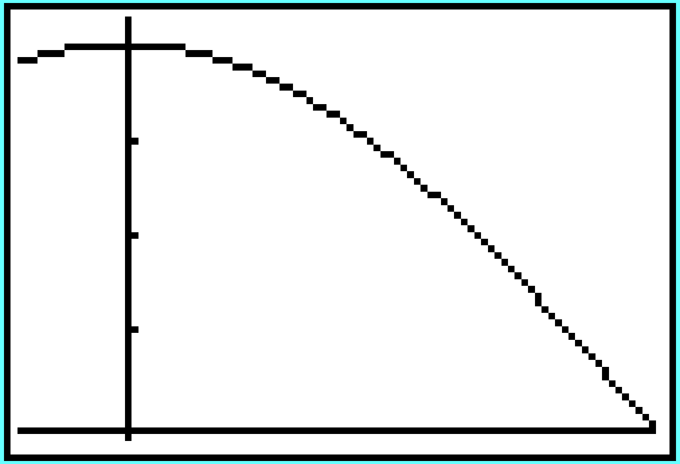


Determine what value best approximates the definite integral.

45. $\int_0^4 \sqrt{x}\,dx =$ 5



46. $\int_0^{\frac{1}{2}} 4\cos(\pi x)\,dx =$ 4/3



In exercises 47-52 state if the statement is true or false. If false, explain why or give a counterexample

47. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ **True: one of properties**

48. $\int_a^b [f(x)g(x)]dx = \left[\int_a^b f(x)dx\right]\left[\int_a^b g(x)dx\right]$

False $\text{CE}: \int_0^1 \left[x\sqrt{x}\right]dx \neq \left[\int_a^b xdx\right]\left[\int_a^b \sqrt{x}dx\right]$ $\frac{2}{5} \neq \frac{1}{2} \times \frac{2}{3}$

49. If the norm of a partition approaches zero, then the number of subintervals approaches infinity. True

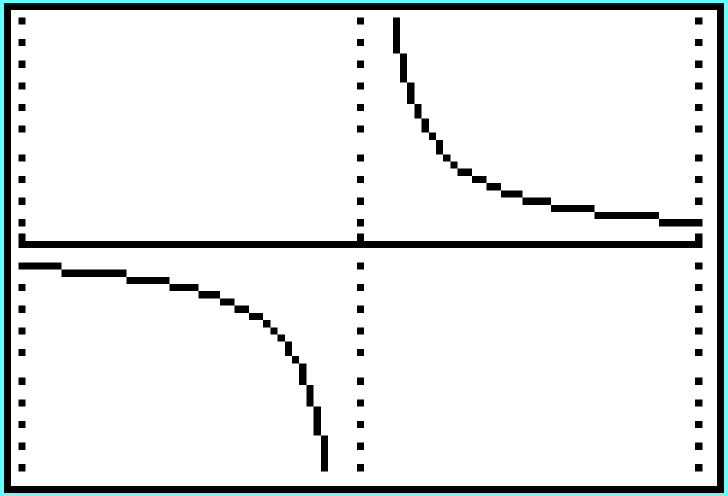
50. If f is increasing on the interval [a, b], then the minimum value of f(x) on [a, b] is f(a). True

51. The value of $\int_a^b f(x)dx$ must be positive. False

52. If $\int_a^b f(x)dx > 0$ then f is non-negative for all x in [a, b]. False

57. Determine if the function below is integrable on the interval [3, 5]. Explain.

$$f(x) = \frac{1}{x - 4}$$



No. The discontinuity at x = 4 results in an infinite area.

58. Determine if the following function is integrable on the interval [0, 1]. Explain.

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

No. There are an infinite number of rational and irrational numbers on any interval regardless of the length of the interval.

60. Evaluate, if possible, the integral: $\int_0^2 [\![x]\!]\,dx$