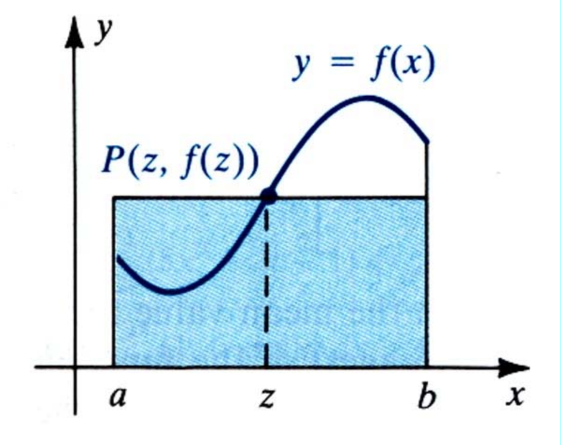
Mean Value Theorem for Integrals

$$\int_a^b f(x)dx = f(z)(b-a)$$



$$f(z) = \frac{\int_a^b f(x)dx}{(b-a)}$$

Ex: Find the value of x on the interval [a,b] which satisfies the MVT if:

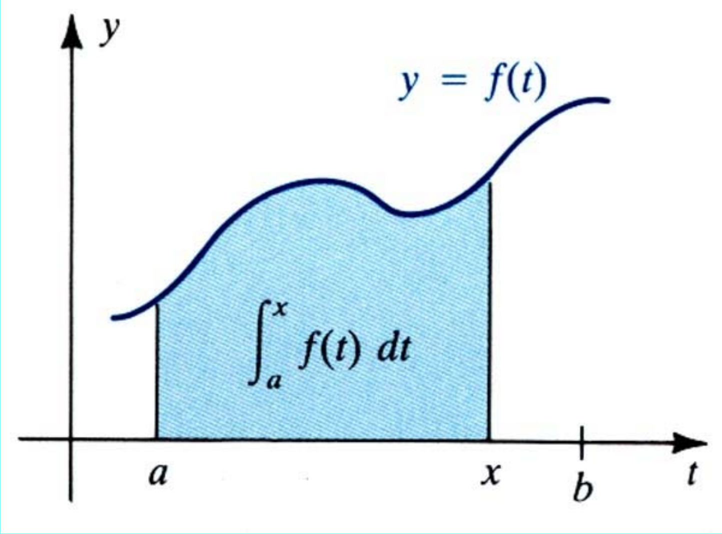
$$\int_0^3 x^2 dx = 9$$

$$z^2(3-0) = 9$$

$$z^2 = 3 \rightarrow z = \sqrt{3}$$

Fundamental Theorem of Calculus I

To avoid confusion use t as the independent variable. If f(t) is integrable on [a,b] and a<x<b, then define a function g(x) which represents the area under f(t):



$$g(x) = \int_a^x f(t)dt \quad \text{and} \quad g'(x) = f(x)$$

States that the there is an area function which gives the area under the graph of a function and it is the antiderivative of that function.

Fundamental Theorem of Calculus

Suppose f(x) is continuous on the closed interval [a,b].

Part I: If the function G(x) is defined by

$$G(x) = \int_a^x f(t)\,dt$$

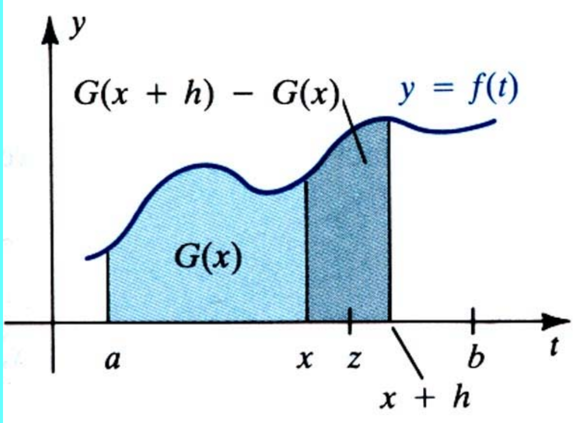
For every x in [a,b], then G is an antiderivative of f on [a,b]

Part II: If F is any antiderivative of f on [a,b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

PROOF Part I

Must show $$\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}=f(x)$$



$$\mathrm{G(x+h)\text{-}G(x)}=\int_{\mathrm{a}}^{\mathrm{x+h}}\mathrm{f(t)dt}-\int_{a}^{x} f(t)dt$$

$$=\int_{\mathrm{a}}^{\mathrm{x+h}}\mathrm{f(t)dt}+\int_{x}^{a} f(t)dt$$

$$=\int_{\mathrm{x}}^{\mathrm{x+h}}\mathrm{f(t)dt}$$

By MVT: $$\int_{\mathrm{x}}^{\mathrm{x+h}}\mathrm{f(t)dt}= f(z)h$$

$$\frac{G(x+h)-G(x)}{h}=\frac{1}{h}\int_{\mathrm{x}}^{\mathrm{x+h}}\mathrm{f(t)dt}= f(z)$$

$$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\int_{\mathrm{x}}^{\mathrm{x+h}} \mathrm{f(t)dt} = f(z)$$

$$\frac{G(x+h)-G(x)}{h} = f(z)$$

Since x<z<x+h it follows that: $\underset{h\to 0}{Limit}\ f(z) = \underset{z\to x}{Limit}\ f(z) = f(x)$

$$\underset{h\to 0}{Limit}\frac{G(x+h)-G(x)}{h} = \underset{h\to 0}{Limit}\ f(z) = f(x)$$

$$\underset{h\to 0}{Limit}\frac{G(x+h)-G(x)}{h} = f(x)$$

The derivative of the area function is f(x).

The antiderivative of f(x) is the area function of f(x).

Part II: Prove $\int_a^b f(x)\,dx = F(b) - F(a)$

Let F be any antiderivative of f and let G be the special antiderivative, the area function. We know that there is a constants C such that: $G(x) = F(x) + C$

Hence: $$\int_a^x f(t)\,dt = F(x) + C$$

Let x=a: $$\int_a^a f(t)\,dt = 0 = F(a) + C \rightarrow C = -F(a)$$

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

Since this is an identity for all values of x on [a,b]

$$\int_a^b f(t)\,dt = F(b) - F(a) \quad \textbf{or} \quad \int_a^b f(x)\,dx = F(b) - F(a)$$

Ex: Find the area of the region between y=sin x and the x-axis from x=0 to x=□.

$$Area = \int_0^{\pi} \sin x = -\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) + 1 = 2$$

Ex: Evaluate $\displaystyle\int_{-1}^{2} (x^3+1)^2 dx$

$$= \int_{-1}^{2} (x^6 + 2x^3 + 1) dx$$

$$= \left[\frac{x^7}{7} + 2\frac{x^4}{4} + x \right]_{-1}^{2}$$

$$= \left[\frac{2^7}{7} + 2\frac{2^4}{4} + 2 \right] - \left[\frac{(-1)^7}{7} + 2\frac{(-1)^4}{4} - 1 \right] = \frac{405}{14}$$

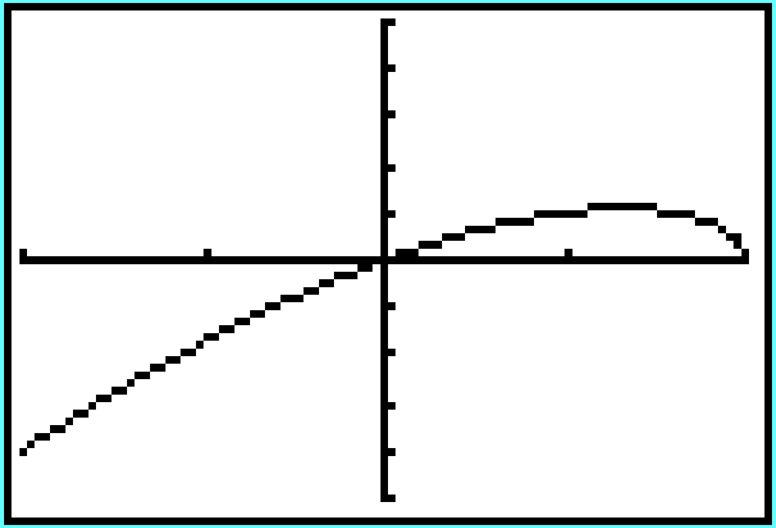
Ex: Evaluate $\displaystyle\int_1^4 \left(5x - 2\sqrt{x} + \frac{32}{x^3}\right) dx$

$$\int_1^4 \left(5x - 2x^{\frac{1}{2}} + 32x^{-3}\right) dx$$

$$\left[\frac{5}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}} - 16\frac{1}{x^2}\right]_1^4$$

$$\left[\frac{5}{2}(4)^2 - \frac{4}{3}(4)^{\frac{3}{2}} - \frac{1}{16}(16)\right] - \left[\frac{5}{2}(1)^2 - \frac{4}{3}(1)^{\frac{3}{2}} - \frac{1}{16}(1)\right] = \frac{259}{6}$$

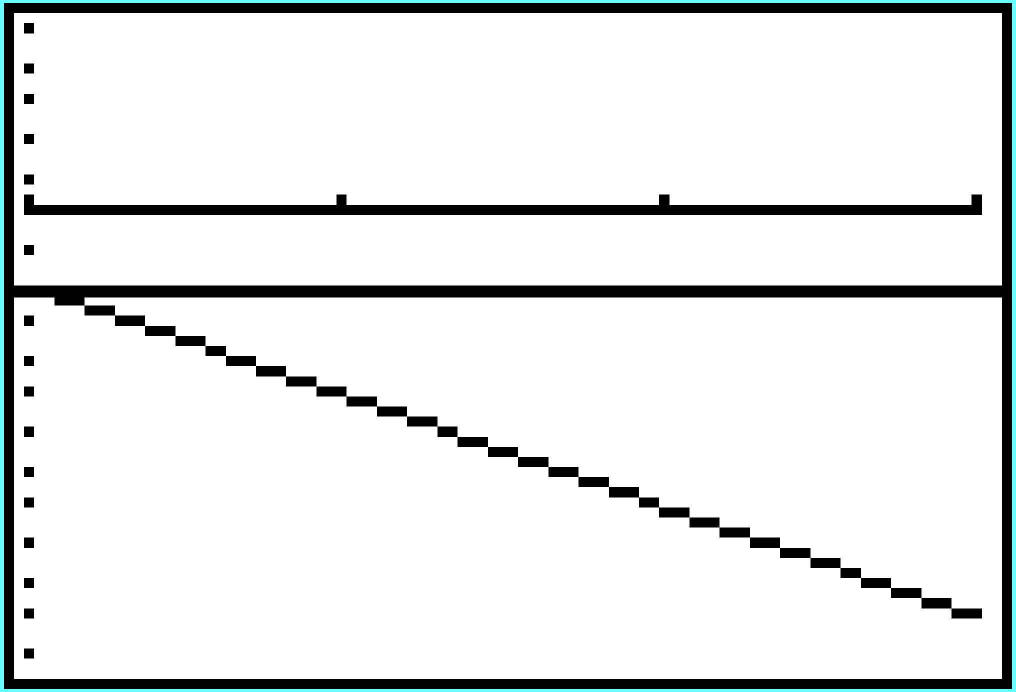
4. Use the graph to determine if the following definite integral is positive, negative or zero.



Negative

Evaluate the following definite integrals and use a graphing utility to verify results.

8. $\displaystyle\int_2^5 (-3v+4)\,dv = \left[\frac{-3v^2}{2}+4v\right]_2^5 = \left[\frac{-3(5)^2}{2}+4(5)\right]-\left[\frac{-3(2)^2}{2}+4(2)\right] = -\frac{39}{2}$



$$\frac{1}{2}3(9)+3(2)=\frac{27}{2}+\frac{12}{2}=\frac{39}{2}$$

16. $$\int_{-3}^{3} v^{\frac{1}{3}}dv = \left[\frac{3}{4}v^{\frac{4}{3}}\right]_{-3}^{3} = \left[\frac{3}{4}(3)^{\frac{4}{3}}\right] - \left[\frac{3}{4}(-3)^{\frac{4}{3}}\right] = 0$$

20. $$\int_{0}^{2}(2-t)\sqrt{t}dt = \int_{0}^{2}\left(2t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)dt = \left[\frac{4t^{\frac{3}{2}}}{3} - \frac{2t^{\frac{5}{2}}}{5}\right]_{0}^{2}$$

$$= \left[\frac{4(2)^{\frac{3}{2}}}{3} - \frac{2(2)^{\frac{5}{2}}}{5}\right] = \left[\frac{8(2)^{\frac{1}{2}}}{3} - \frac{8(2)^{\frac{1}{2}}}{5}\right] = \frac{40\sqrt{2} - 24\sqrt{2}}{15} = \frac{16\sqrt{2}}{15}$$

24. $$\int_{0}^{4}\left|x^2 - 4x + 3\right|dx = \int_{0}^{4}\left|\left(x^2 - 4x + 4\right) + 3 - 4\right|dx = \int_{0}^{4}\left|(x-2)^2 - 1\right|dx$$

$(x-2)^2 - 1 = 0 \text{ @ } x = \{1, 3\}$ $\quad$ $x < 0$ on $(1, 3)$

$$\int_{0}^{1}\left(x^2 - 4x + 3\right)dx - \int_{1}^{3}\left(x^2 - 4x + 3\right)dx + \int_{3}^{4}\left(x^2 - 4x + 3\right)dx$$

$$\int_0^1 \left(x^2 - 4x + 3\right)dx - \int_1^3 \left(x^2 - 4x + 3\right)dx + \int_3^4 \left(x^2 - 4x + 3\right)dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left[\frac{(1)^3}{3} - 2(1)^2 + 3(1)\right] - \left\{\left[\frac{(3)^3}{3} - 2(3)^2 + 3(3)\right] - \left[\frac{(1)^3}{3} - 2(1)^2 + 3(1)\right]\right\}$$

$$+ \left[\frac{(4)^3}{3} - 2(4)^2 + 3(4)\right] - \left[\frac{(3)^3}{3} - 2(3)^2 + 3(3)\right]$$

$$= 2\left[\frac{(1)^3}{3} - 2(1)^2 + 3(1)\right]_0^1 + \left[\frac{(4)^3}{3} - 2(4)^2 + 3(4)\right]_3^4 - 2\left[\frac{(3)^3}{3} - 2(3)^2 + 3(3)\right]$$

$$= 2\left(\frac{4}{3}\right) + \frac{4}{3} - 2(0) = \frac{12}{3} = 4$$

28. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2-\csc^2 x\right)dx=\left[2x-\cot x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\left[2\left(\frac{\pi}{2}\right)-\cot\left(\frac{\pi}{2}\right)\right]-\left[2\frac{\pi}{4}-\cot\frac{\pi}{4}\right]$$

$$=\left(\pi+0\right)\left(\frac{\pi}{2}+1\right)=\frac{\pi-2}{2}$$

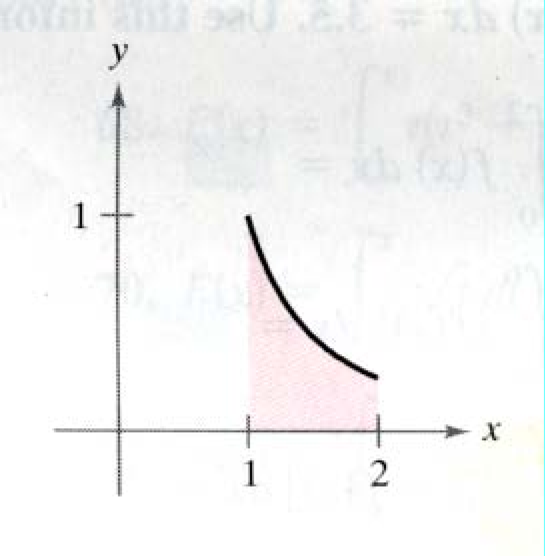
32. Buffon's Needle Experiment: A horizontal plane is ruled with parallel lines 2 inches apart. If a 2-inch needle is tossed randomly onto the plane, the probability that the needle will touch a line is

$$P=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin\theta\, d\theta$$

Where θ is the acute angle between the needle and any one of the parallel lines. Find this probability.

$$P=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin\theta\, d\theta=\frac{2}{\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{2}}=\frac{2}{\pi}\left[0-1\right]=\frac{2}{\pi}=.637$$

36. $\int_1^2 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = -\frac{1}{2} + 1 = \frac{1}{2}$



40. Find the area of the region bound by:

$y = 1 + \sqrt{x}$

$x = 0$

$x = 4$

$y = 0$

$$\int_0^4 1 + \sqrt{x} = \left[x + \frac{2x^{\frac{3}{2}}}{3}\right]_0^4 = \left[4 + \frac{2(4)^{\frac{3}{2}}}{3}\right] = 4 + \frac{2}{3}8 = \frac{28}{3}$$

44. Find the value of c guaranteed by the MVT over the interval.

$$f(x)=\frac{9}{x^3} \quad [1,3]$$

$$\int_1^3 \frac{9}{x^3} = \left[-\frac{9}{2x^2}\right]_1^3 = \left[-\frac{9}{2(3)^2}\right]-\left[-\frac{9}{2(1)^2}\right] = -\frac{1}{2}+\frac{9}{2} = 4$$

By MVT:

$$\frac{9}{c^3}(2)=4 \rightarrow \frac{9}{c^3}=2 \rightarrow c^3=\frac{9}{2} \rightarrow c=\sqrt[3]{\frac{9}{2}}=1.65$$

48. Graph the function over the interval. Find the average value of the function over the interval and find all values of x in the interval for which the function equals its average value.

$$f(x) = \frac{x^2+1}{x^2} = 1 + \frac{1}{x^2} \quad \left[\frac{1}{2}, 2\right]$$

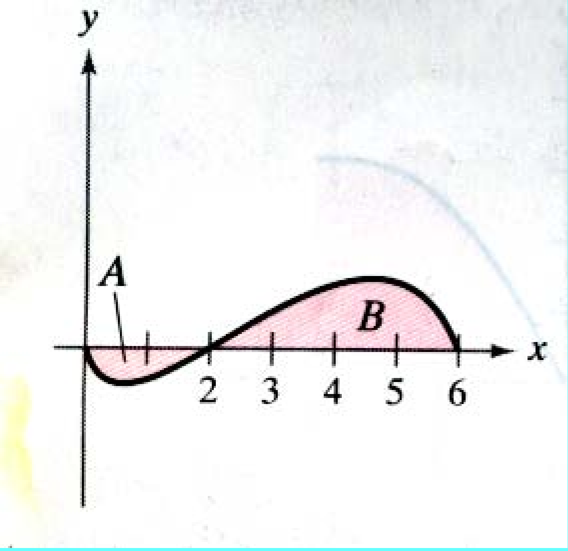
What is "average value of the function"?

Sum of all of the f(x)'s divided by the interval.

$$\frac{\int_{\frac{1}{2}}^{2} \frac{x^2+1}{x^2}}{\left(2-\frac{1}{2}\right)} = \frac{1}{\left(2-\frac{1}{2}\right)}\int_{\frac{1}{2}}^{2} \frac{x^2+1}{x^2} = \frac{2}{3}\left[x - \frac{1}{x}\right]_{\frac{1}{2}}^{2} = \frac{2}{3}\left[2-\frac{1}{2}\right] - \frac{2}{3}\left[\frac{1}{2}-2\right] = 2$$

52. Use the graph of f to the right and fill in the blanks, if region A has area of 1.5 and $\int_0^6 f(x)dx = 3.5$

$$\int_2^6 f(x)dx = \underline{\quad 5 \quad}$$



60. **The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is v == k(R^2-r^2) where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery.**

$$\frac{1}{R}\int_0^R k(R^2 - r^2)dr = \frac{1}{R}\left[k(R^2 r - \frac{r^3}{3})\right]_0^R$$

$$= \frac{1}{R}\left[k(R^2 (R) - \frac{(R)^3}{3})\right]$$

$$= kdR^2 - \frac{kR^2}{3} = \frac{2kR^2}{3}$$

61. The force F of a hydraulic cylinder in a press is proportional to the square of sec x, where x is the distance in meters that the cylinder is extended in its cycle. The domain of F is [0, π/3] and F(0) = 500.

(a) Find F as a function of x.

$$F(x) = k\sec^2 x$$

$$F(0) = 500 = k\sec^2 0 = k$$

$$F(x) = 500\sec^2 x$$

(b) Find the average force exerted by the press over the interval [0, π/3].

$$\frac{1}{\frac{\pi}{3} - 0}\int_0^{\frac{\pi}{3}} 500\sec^2 x\,dx = \frac{1500}{\pi}\left[\tan x\right]_0^{\frac{\pi}{3}} = \frac{1500}{\pi}\left[\tan\frac{\pi}{3}\right] - \frac{1500}{\pi}\left[\tan 0\right]$$

$$= \frac{1500\sqrt{3}}{\pi} = 827$$

62. The volume V in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$, where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

$$\frac{1}{5}\int_0^5 \left(0.1729t + 0.1522t^2 - 0.0374t^3\right)dt$$

$$= \frac{1}{5}\left[\frac{0.1729t^2}{2} + \frac{0.1522t^3}{3} - \frac{0.0374t^4}{4}\right]_0^5 = .53$$