Integration

by

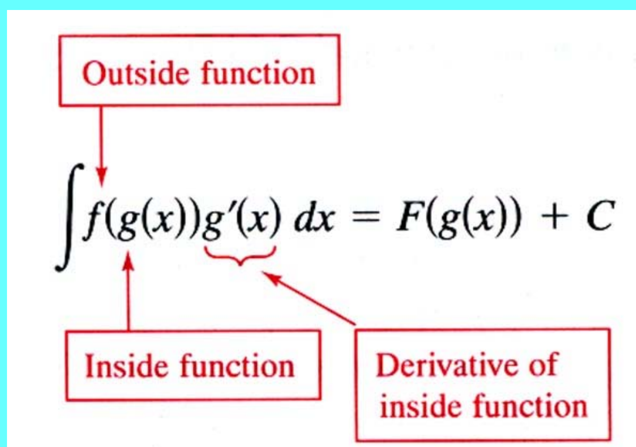
Substitution

Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)dx$

Follows that: $\int f(g(x))g'(x)dx = F(g(x)) + C$

Problem is when you have $\int f(x)g(x)dx$

deciding which is the inner and which is the outer function and identifying the derivative of the inner function.



Ex: $\int (x^2-1)^3 2x\,dx$

u^3 is the outer function

(x^2-1) is the inner function

2x is the derivative of inner

Ex: $\int 2x(x^2+1)^2 dx = \dfrac{(x^2+1)^3}{3} + C$

Missing constants can be introduced as a unique form of 1.

Ex: $\int x(x^2+1)^2 dx = \dfrac{1}{2}\int 2x(x^2+1)^2 dx = \dfrac{(x^2+1)^3}{3} + C$

Variables can be changed to simplify the expression.

If we let $u = x^2 + 1$ **and** $du = 2x\,dx$ **then**

$$\int x(x^2+1)^2 dx = \frac{1}{2}\int (x^2+1)^2(2x\,dx) = \frac{1}{2}\int u^2 du = \frac{u^3}{6} + C$$

$$\frac{u^3}{6} + C = \frac{\left(x^2+1\right)^3}{6} + C$$

Ex: $$\int (3x-1)^4 dx = \frac{1}{3}\int 3(3x-1)^4 dx = \frac{(3x-1)^5}{15} + C$$

Ex: $$\int 3x^2\left(x^3-1\right)^{\frac{1}{2}} dx = \frac{\left(x^3-1\right)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\left(x^3-1\right)^{\frac{3}{2}}}{3} + C$$

Ex: $$\int \frac{-4x}{\left(1-2x^2\right)^2} dx = \int \left(1-2x^2\right)^{-2}(-4x)\,dx = \frac{\left(1-2x^2\right)^{-1}}{-1} + C$$

Ex: $$\int \cos^2 x \sin x\,dx = -\int \cos^2 x(-\sin x)\,dx = -\frac{\cos^3 x}{3} + C$$

4. Identify u and du for: $\int \sec(2x)\tan(2x)\,dx$

u=____2x______

du=___2dx_____

8. Evaluate: $\int (x^2-1)^3(2x)\,dx = \frac{(x^2-1)^4}{4} + C$

12. Evaluate: $\int x(4x^2+3)^3\,dx = \frac{1}{8}\int (4x^2+3)^3\,8x\,dx = \frac{1}{8}\left[\frac{(4x^2+3)^4}{4}\right] + C$

16. Evaluate: $\int \frac{x^2}{(16-x^3)^2}\,dx = -\frac{1}{3}\int \frac{-3x^2}{(16-x^3)^2}\,dx = \frac{1}{3}\left(\frac{1}{(16-x^3)}\right)$

20. $$\int \frac{1}{2\sqrt{x}}dx = \int x^{-\frac{1}{2}}\left(\frac{1}{2}\right)dx = \int \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\left(\frac{1}{2}\right)dx = \sqrt{x} + C$$

24. $$\int \left(\frac{t^3}{3} + \frac{1}{4t^2}\right)dt = \frac{t^4}{12} - \frac{1}{4t} + C$$

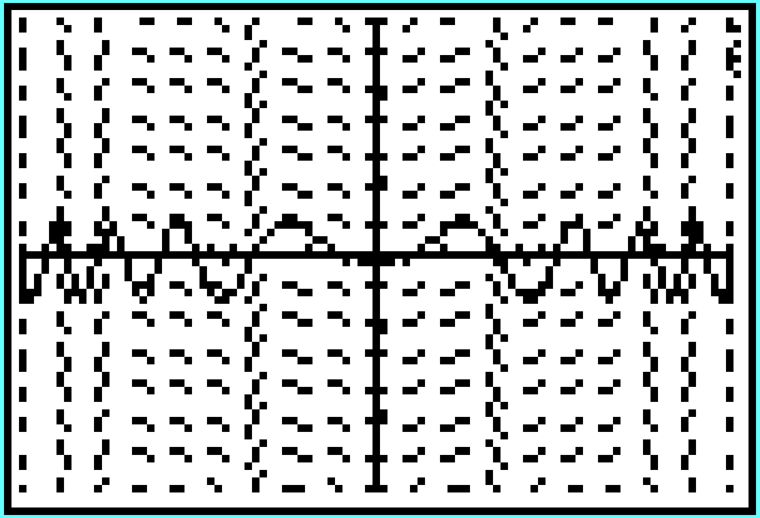
28. **Solve:** $$\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$$

$$\int \frac{10x^2}{\sqrt{1+x^3}}dx = \frac{10}{3}\int \frac{1}{\sqrt{1+x^3}}3x^2dx = \frac{10}{3}\frac{\sqrt{1+x^3}}{\frac{1}{2}} + C = \frac{20}{3}\sqrt{1+x^3} + C$$

32. A direction field is a graph showing the slopes at a series of points. Slopes are given by a differential equation $\frac{dy}{dx} = f(x)$

Direction fields give the outline of a family of equations which are the integrals of the differential equation. Sketch a solution to the differential equation given.

$$\frac{dy}{dx} = x\cos x^2$$



36. $$\int \cos(6x)\,dx = \frac{1}{6}\int \cos(6x)\,6dx = \frac{1}{6}\sin 6x + C$$

40. $$\int \sqrt{\cot x}\,\csc^2 x\,dx = -\int (\cot x)^{\frac{1}{2}}\left(-\csc^2 x\right)dx = -\frac{2}{3}(\cot x)^{\frac{3}{2}} + C$$

44. $$\int \csc^2\left(\frac{x}{2}\right)dx = -2\int -\csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = -2\cot\left(\frac{x}{2}\right) + C$$

48. $$\int x\sqrt{2x+1}\,dx$$ Let $u = 2x+1,\quad x = \frac{u-1}{2}$ and $dx = \frac{1}{2}du$

$$\int x\sqrt{2x+1}\,dx = \int \left[\frac{1}{2}(u-1)\right]\left[\sqrt{u}\right]\left[\frac{1}{2}du\right]$$

$$= \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{4}\left(\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right) + C = \frac{2(2x+1)^{\frac{5}{2}}}{10} - \frac{2(2x+1)^{\frac{3}{2}}}{12} + C$$

52. $\displaystyle\int \frac{2x-1}{\sqrt{x+3}}dx$ Let: $u = x+3, \quad x = u-3 \quad \text{and} \quad dx = du$

$$\int \frac{2x-1}{\sqrt{x+3}}dx = \int \frac{2(u-3)-1}{\sqrt{u}}dx = \int \frac{2u-7}{\sqrt{u}}du = \int \left(2u^{\frac{1}{2}} - 7u^{-\frac{1}{2}}\right)du$$

$$= \frac{4u^{\frac{3}{2}}}{3} + 14u^{\frac{1}{2}} + C$$

56. $\displaystyle\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{2}\int_0^1 \sqrt{1-x^2}(-2x)dx = -\frac{1}{3}\left[\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{\frac{3}{2}}\right] - \frac{1}{3}\left[\left(1-(0)^2\right)^{\frac{3}{2}}\right] = \frac{1}{3}$$

60. $$\int_0^2 x\sqrt[3]{4+x^2}\,dx=\frac{1}{2}\int_0^2 2x\sqrt[3]{4+x^2}\,dx=\frac{1}{2}\frac{\left(4+x^2\right)^{\frac{4}{3}}}{\frac{4}{3}}=\left[\frac{3}{8}\left(4+x^2\right)^{\frac{4}{3}}\right]_0^2=3.62$$

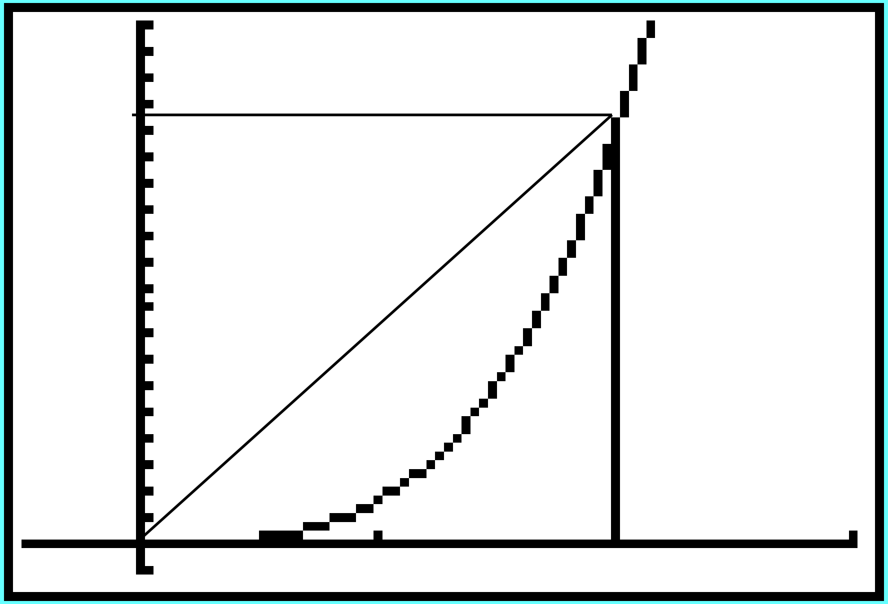
64. $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(x+\cos x)\,dx=\left[\frac{x^2}{2}+\sin x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}=\frac{5\pi^2}{72}+\frac{2-\sqrt{3}}{2}$$

68. $$\text{Area}=\int_0^{\pi}(\sin x+\cos 2x)\,dx=-\int_0^{\pi}-\sin x\,dx+\frac{1}{2}\int_0^{\pi}(\cos 2x)2dx$$

$$=\left[-\cos x+\frac{1}{2}\sin 2x\right]_0^{\pi}=2$$

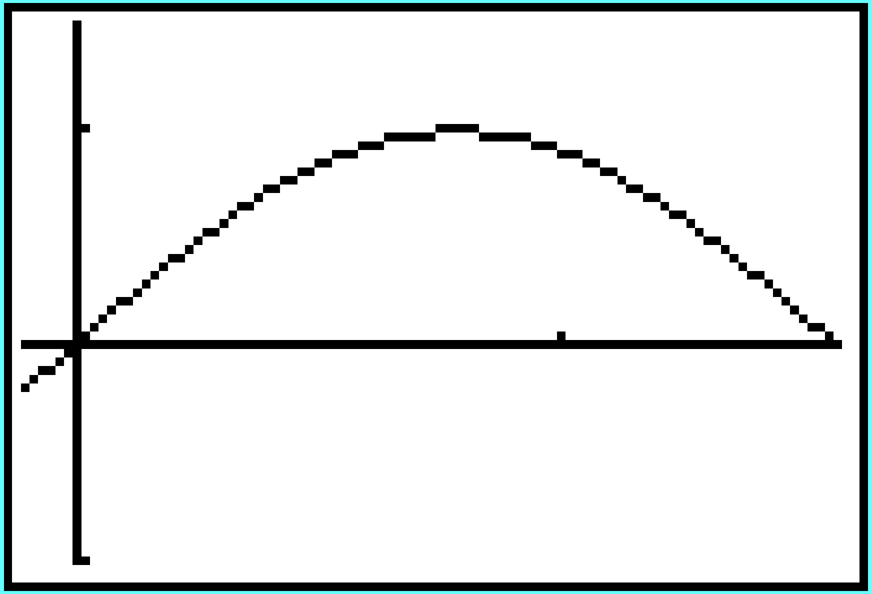
72. **Area** $= \int_0^2 x^3\sqrt{x+2}\,dx$

=7.58



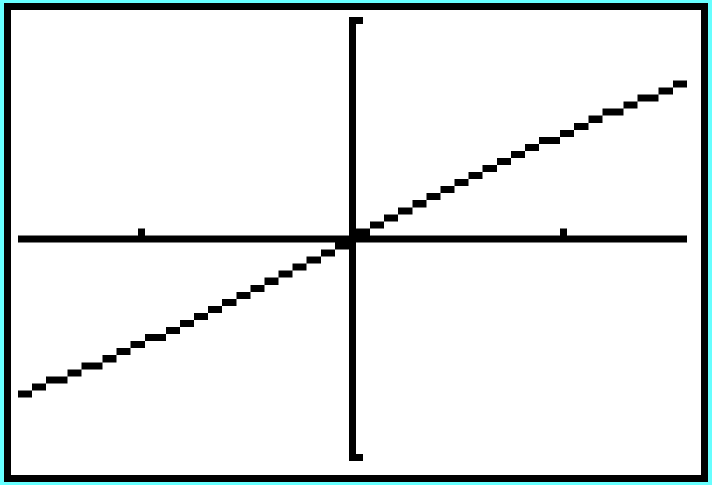
76. **Area** $= \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

= 1



80. Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each of the integrals.

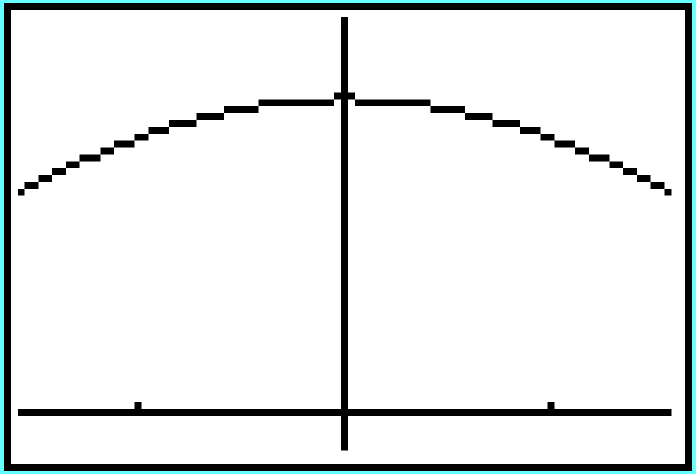
(a) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin x \, dx$



(b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx$

$\displaystyle = 2\int_{0}^{\frac{\pi}{4}} \cos x \, dx$

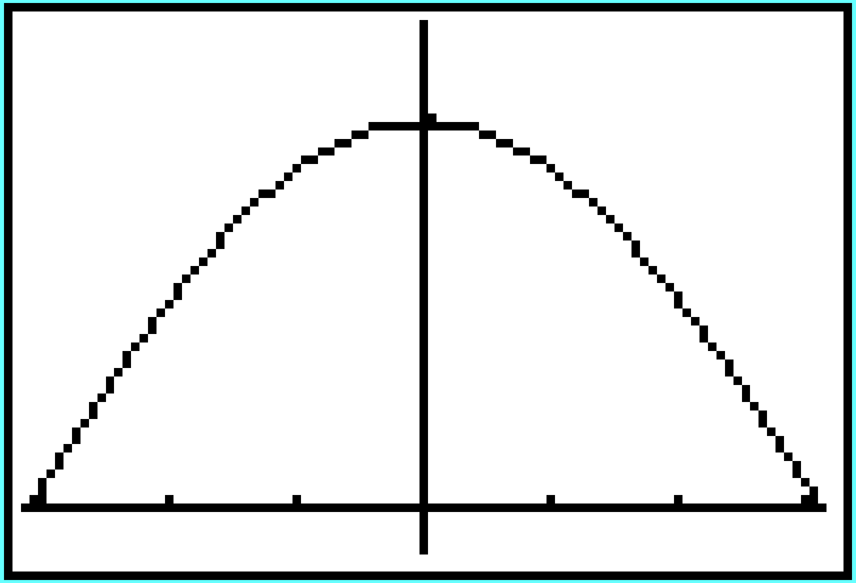
$= \sqrt{2}$



(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$

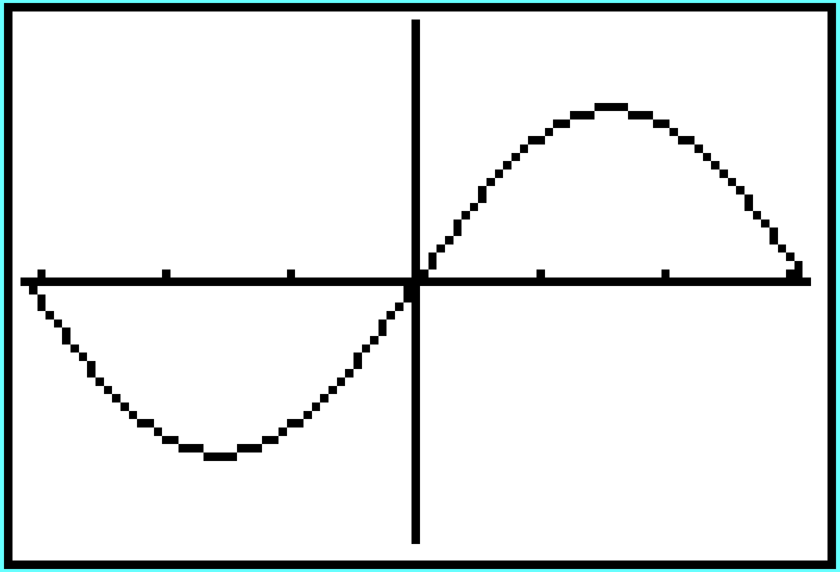
$= 2\int_{0}^{\frac{\pi}{2}} \cos x \, dx$

$= 2$



(d) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x \, dx$

$= 0$



84. The rate of disbursement dQ/dt of a 2 million dollar federal grant is proportional to the square of 100 - t. Time t is measured in days ($0 \le t \le 100$), and Q is the amount that remains to be disbursed. Find the amount that remains to be disbursed after 50 days. Assume that all the money will be disbursed in 100 days

$$\frac{dQ}{dt} = k(100-t)^2 \qquad Q(t) = \int k(100-t)^2\,dt = -\frac{k(100-t)^3}{3} + C$$

$$Q(100) = C = 0$$

$$Q(t) = -\frac{k(100-t)^3}{3}$$

$$Q(0) = -\frac{k(100-0)^3}{3} = 2{,}000{,}000 \rightarrow k = -6$$

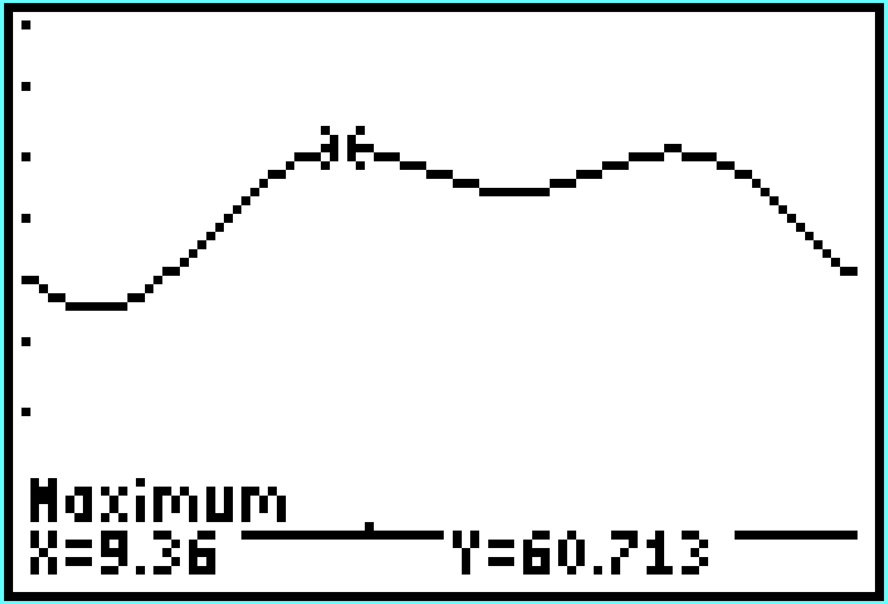
$$Q(t) = \frac{6(100-t)^3}{3}$$

$$Q(50) = \frac{6(50)^3}{3} = 250{,}000$$

88. A model for the flow rate of water at a pumping station on a given day is $R(t) = 52 + 7\sin\left(\frac{\pi t}{6} + 3.6\right) + 9\cos\left(\frac{\pi t}{12} + 8.9\right)$

Where $0 \le t \le 24$. R is the flow rate in thousands of gallons per hour, and t is the time in hours.

(a) Use a graphing utility to graph the rate function and approximate the maximum flow rate.



(b) Approximate the total volume pumped in a single day.

$$\text{Volume} = \int_0^{24} \left(52 + 7\sin\left(\frac{\pi t}{6} + 3.6\right) + 9\cos\left(\frac{\pi t}{12} + 8.9\right)\right) dt$$

$$= \int_0^{24} 52 dt + 7\int_0^{24} \sin\left(\frac{\pi t}{6} + 3.6\right) dt + 9\int_0^{24} \cos\left(\frac{\pi t}{12} + 8.9\right) dt$$

$$= \left[52t - \frac{42}{\pi}\cos\left(\frac{\pi t}{6} + 3.6\right) + \frac{108}{\pi}\sin\left(\frac{\pi t}{12} + 8.9\right)\right]_0^{24} = 1272$$

92. T/F: $\int x\left(x^2+1\right)dx = \frac{1}{2}\left(\frac{1}{3}x^3+x\right)+C$

$$= \int x\left(x^2+1\right)dx = \int x^3dx + \int xdx \quad = \frac{x^4}{4}+x+C$$

False

96. T/F: $\int \sin^2 2x \cos 2x \, dx = \frac{1}{3}\sin^3 2x + C$

$$= \frac{1}{2}\int \sin^2(2x)\left(\cos 2x \, 2\right)dx$$

$$= \frac{1}{2}\frac{\sin^3 2x}{3} + C = \frac{1}{6}\sin^3 2x + C$$

False