Numerical Integration

Riemann Sums

Right

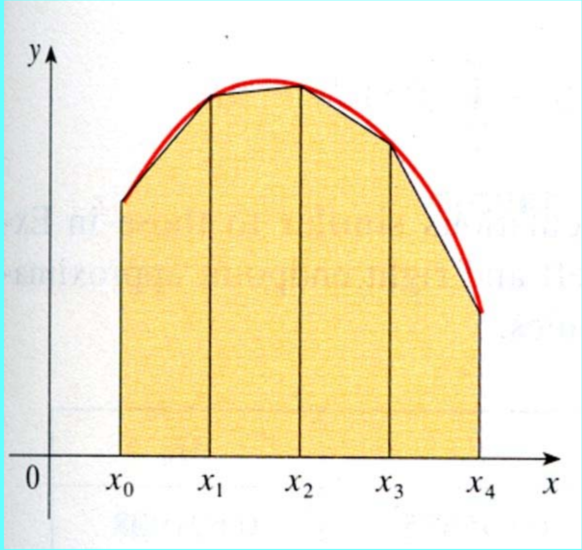
Left

Midpoint

Trapezoid Rule

Simpson's Rule

<u>Trapezoid Rule</u>: Uses trapezoids to estimate area rather than rectangles.



$$Area = \frac{(b_1 + b_2)}{2} h$$

$$= \frac{(f(x_i) + f(x_{i+1}))}{2} \Delta x$$

$$= \frac{(f(x_0) + f(x_1))}{2} \Delta x + \frac{(f(x_1) + f(x_2))}{2} \Delta x + \ldots$$

$$= \frac{\Delta x}{2} \left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \ldots \right]$$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2} \left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + f(x_n) \right]$$

Ex: Use the trapezoidal rule with n=5 to find an approximation of the area bounded by f(x) =1/x, the x-axis, x=1 and x=2.

$$\int_1^2 \frac{1}{x}dx \approx \frac{0}{2}\left[\frac{1.}{1}+\frac{22}{1.2}+\frac{2}{1.4}+\frac{2}{1.6}+\frac{2}{1.8}+\frac{2}{2}\right]$$

$$\approx 0.695635$$

Ex: Use the trapezoidal rule with n=10 to find an approximation of the area of:

$$\int_0^1 e^{x^2}dx \approx \frac{.1}{2}\left[e^0+2(e^{.01})+2(e^{.04})+\ldots+2(e^{.81})+e\right]$$

$$\approx .05\left[1+2.02+2.08+\ldots+4.50+2.72\right]$$

$$\approx 1.46717$$

Ex: Use the trapezoidal rule with n=10 to find an approximation of the area of: $\int_0^1 e^{x^2}\,dx$

STAT/Edit

L_1 = Sequence

$L_2 = x_1 + (L_1 \times \Delta x)$

$L_3 = f(L_2)$

L_4 = Coefficients

$L_5 = L_3 \times L_4$

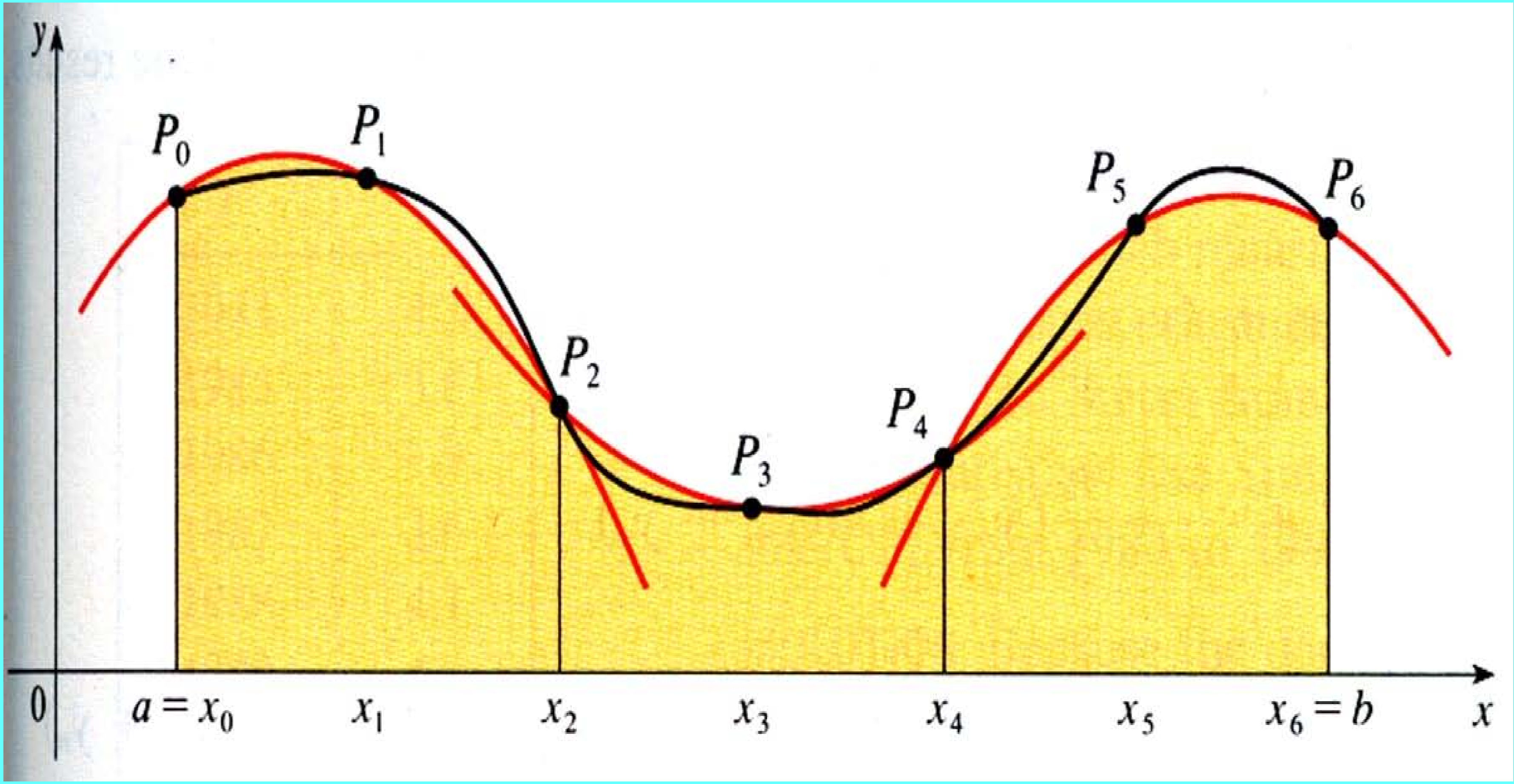
$(\Delta x/2)\text{Sum}(L_5)$

$$\text{Area} = \frac{.1}{2}(29.343) = 1.467$$

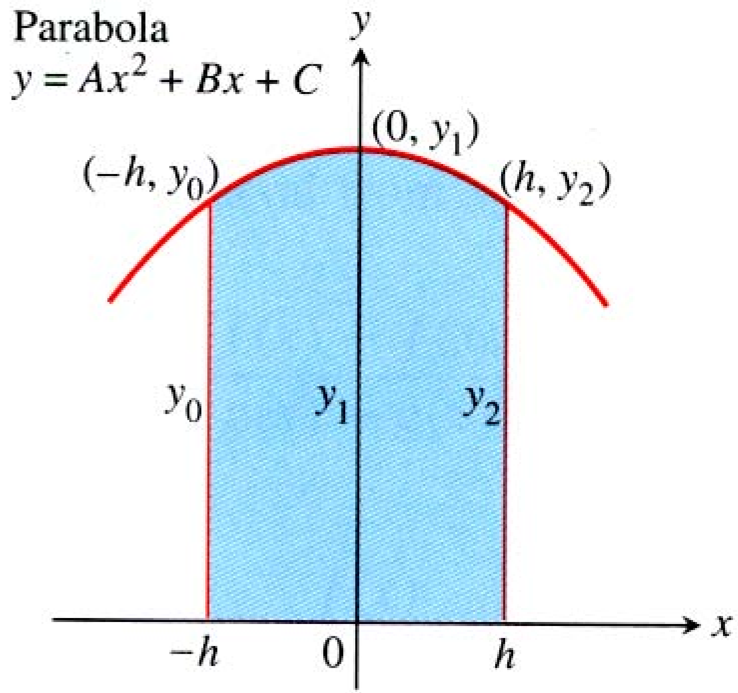
L_1 i	L_2 x_i	L_3 $f(x_i)$	L_4 Coef	L_5 Term
0	0	1	1	1
1	0.1	1.01	2	2.02
2	0.2	1.041	2	2.082
3	0.3	1.094	2	2.188
4	0.4	1.174	2	2.348
5	0.5	1.284	2	2.568
6	0.6	1.433	2	2.866
7	0.7	1.632	2	3.264
8	0.8	1.9	2	3.8
9	0.9	2.248	2	4.496
10	1	2.718	1	2.718
			Sum:	29.343

Simpson's Approximation

Approximation of a curve by using a parabola fitted to three points



Area Under a Parabola:



$$\int_{-h}^{h} \left(Ax^2 + Bx + c\right) dx$$

$$= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}\left(Ah^2 + 6C\right)$$

Since curve passes through (-h, y_0), (0, y_1) and (h, y_2):

$$y_0 = Ah^2 - Bh + C$$
$$y_1 = C$$
$$y_2 = Ah^2 + Bh + C$$

$$C = y_1$$
$$Ah^2 - Bh = y_0 - y_1$$
$$Ah^2 + Bh = y_2 - y_1$$

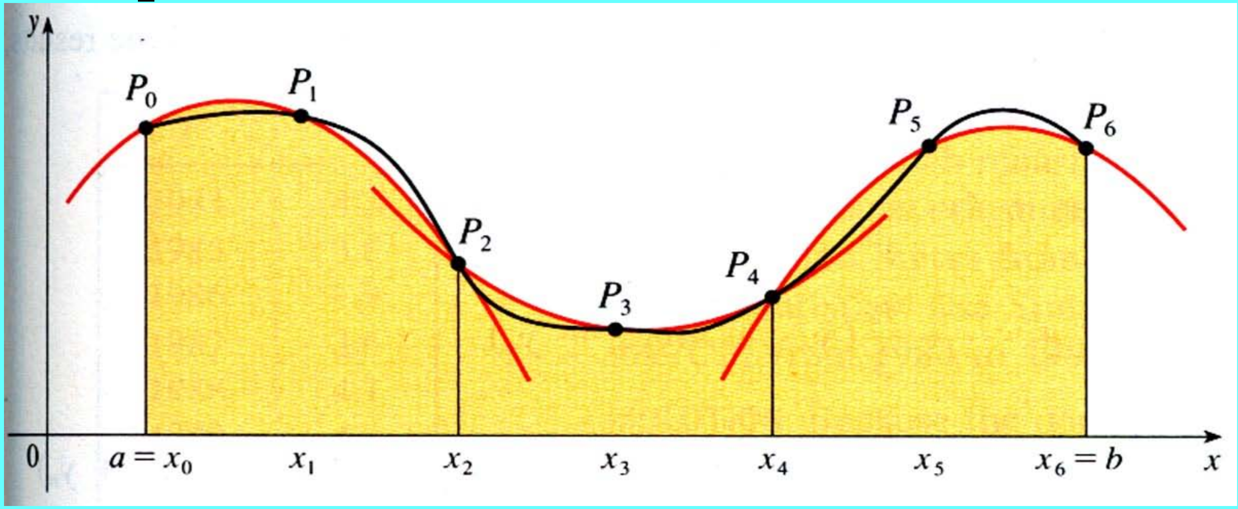
$$2Ah^2 = y_0 + y_2 + 2y_1$$

Substitution into the area expression gives

$$\text{Area} = \frac{h}{3}\left(\left(y_0 - y_2 + 2y_1\right) + 6y_1\right) = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Simpson’s Approximation

Simpson’s Approximation/Rule: Divide [a,b] into an even number of intervals and apply the parabola area formula to successive pairs of intervals.

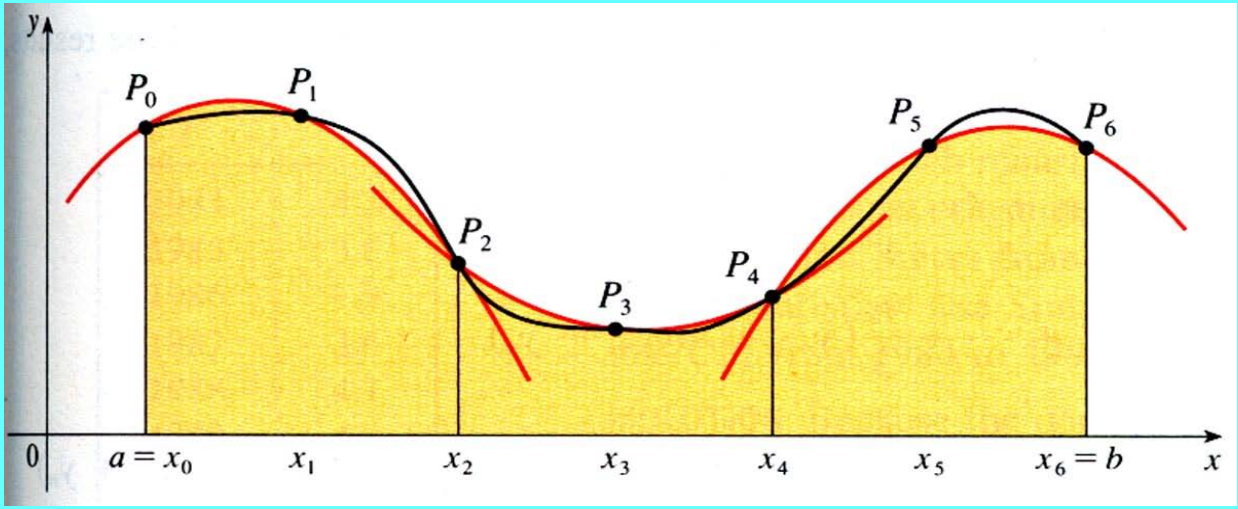


Summing the area under each parabola above gives

$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \dots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

$$= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$



$$\text{Area} = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{3n}\right)\left[y_0 + 4y_1 + 2y_2 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Note the sequence of coefficients: 1,4,2,4,2,4,2,4,…,2,4,1

Ex: Use Simpson's Rule with n=10 to approximate: $\int_1^2 \frac{1}{x}\,dx$

STAT/Edit

L_1 = Sequence

$L_2 = x_1 + L_1(\Delta x)$

$L_3 = f(L_2)$

L_4 = Coefficients

$L_5 = L_3(L_4)$

$(\Delta x/3)$Sum (L_5)

L_1	L_2	L_3	L_4	L_5
I	xi	f(xi)	Coef	Term
0	1	1	1	1
1	1.1	0.9090909	4	3.6363636
2	1.2	0.8333333	2	1.666666
3	1.3	0.7692307	4	3.076923
4	1.4	0.7142857	2	1.428571
5	1.5	0.6666667	4	2.666666
6	1.6	0.625	2	1.25
7	1.7	0.588235	4	2.354921
8	1.8	0.555556	2	1.111111
9	1.9	0.52631578	4	2.105263
10	2	0.5	1	0.5
				20.7945069

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{30}\left(20.794506918\right) = 0.693150231$$

4. $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$

L1	L2	L3 1
0	1	1
1	1.25	.64
2	1.5	.44444
3	1.75	.32653
4	2	.25
------	------	------

L1(1) = 0

L4	L5	L6 6
1	1	
2	1.28	
2	.88889	
2	.65306	
1	.25	
------	------	

L6(1)=

$$\text{Area} = \frac{.25}{2}(4.07195) = .50899$$

L1	L2	L3 1
0	1	1
1	1.25	.64
2	1.5	.44444
3	1.75	.32653
4	2	.25
------	------	------

L1(1) = 0

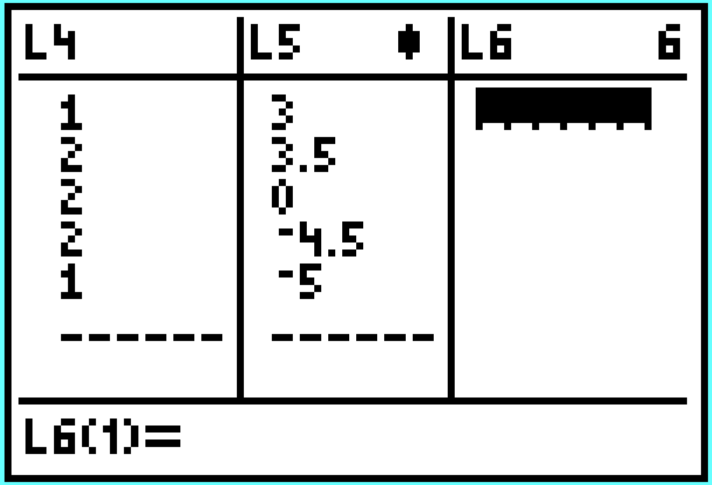
L4	L5	L6 4
1	1	------
4	2.56	
2	.88889	
4	1.3061	
1	.25	
------	------	

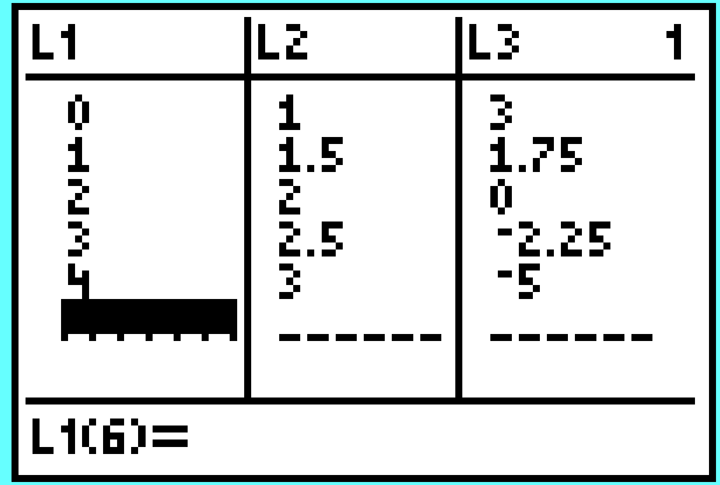
L4(1)=1

$$\text{Area} = \frac{.25}{3}(6.0050) = .5004$$

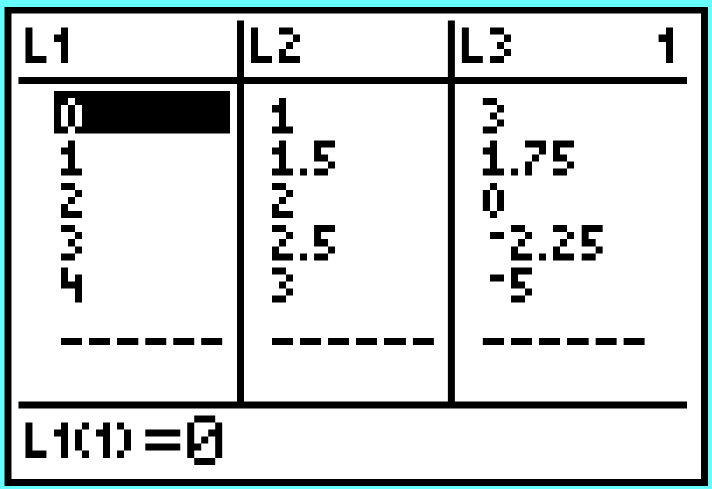
$$\int_1^2 \frac{1}{x^2}\,dx = 0.5000$$

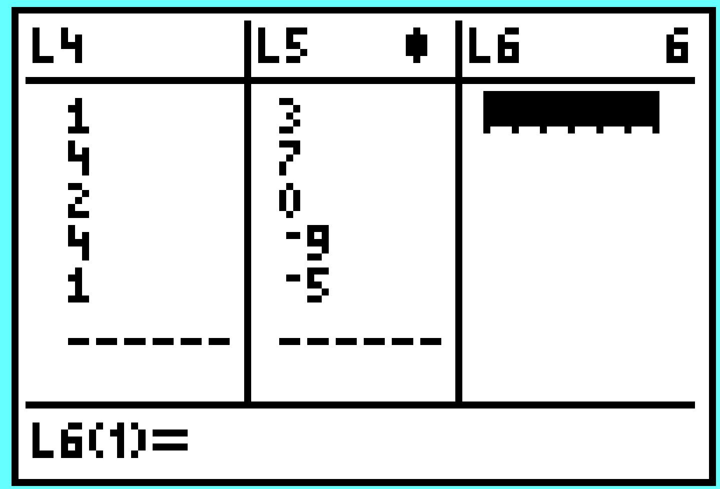
8. $\int_1^3 \left(4 - x^2\right)dx$





$$\text{Area} = \frac{.5}{2}(-3) = -.75$$





$$\text{Area} = \frac{.5}{3}(-4) = -.6667$$

$$\int_1^3 \left(4 - x^2\right)dx = -.6667$$

16. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} \tan x^2\,dx$

L1	L2	L3
0	0	0
1	.22156	.04913
2	.44311	.19891
3	.66467	.47296
4	.88623	1

L1(1) = 0

L4	L5	L6
1	0	
2	.09825	
2	.39782	
2	.94593	
1	1	

L6(1) =

$$\text{Area} = \frac{\pi}{8}(2.44200) = .2705$$

L1	L2	L3
0	0	0
1	.22156	.04913
2	.44311	.19891
3	.66467	.47296
4	.88623	1

L1(1) = 0

L3	L4	L5
0	1	0
.04913	4	.19651
.19891	2	.39782
.47296	4	1.8919
1	1	1

L5(1) = 0

$$\text{Area} = \frac{\pi}{12}(3.4862) = .25746$$

$$\int_0^{\sqrt{\frac{\pi}{4}}} \tan x^2\,dx = 0.256$$

20. $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$

L1	L2	L3	L4	L5	L6
0	0	1	1	1	
1	.7854	.90032	2	1.8006	
2	1.5708	.63662	2	1.2732	
3	2.3562	.3001	2	.60021	
4	3.1416	⁻2E⁻6	1	⁻2E⁻6	
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L1(1)=0 L6(1)=

$$\text{Area} = \frac{\pi}{8}(4.6741) = 1.8355$$

L1	L2	L3	L4	L5	L6
0	0	1	1	1	
1	.7854	.90032	4	3.6013	
2	1.5708	.63662	2	1.2732	
3	2.3562	.3001	4	1.2004	
4	3.1416	⁻2E⁻6	1	⁻2E⁻6	
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L1(1)=0 L6(1)=

$$\text{Area} = \frac{\pi}{12}(7.0749) = 1.8522$$

$$\int_0^{\pi} \frac{\sin x}{x}\,dx = 1.852$$