FOUR

INVERSES OF CIRCULAR AND TRIGONOMETRIC FUNCTIONS

4-1 INVERSE OF A FUNCTION

For every function f a new correspondence can be formed by reversing the pairings between the domain and range elements of f. This new correspondence is called the **inverse** of f and is denoted f^{-1}. For example, consider the function

$$f:\quad 1 \to 0 \qquad 5 \to 3 \qquad -1 \to -2$$

To find f^{-1}, reverse the direction of the mapping.

$$f^{-1}:\quad 0 \to 1 \qquad 3 \to 5 \qquad -2 \to -1$$

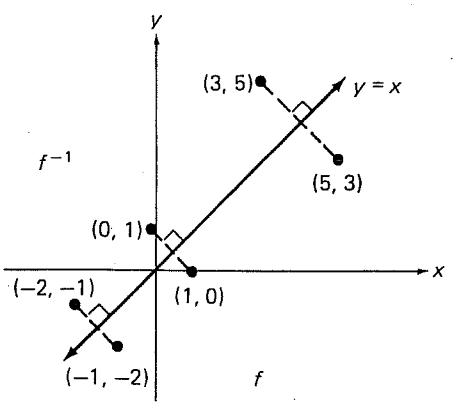


Figure 4-1

The function f and its inverse f^{-1} are graphed in Fig. 4-1. Notice that for each point on the graph of f there is a point on the graph of f^{-1} such that the two points are equidistant from the line $y = x$.

In other words, the graph of f^{-1} is the reflection image of the graph of f across the line $y = x$. This is true for any function f and its inverse since any point (a, b) on the graph of f and the corresponding point (b, a) on the graph of f^{-1} are equidistant from the line $y = x$.

In Example 1, the inverse of a function expressed as an equation is found.

EXAMPLE 1

Find the inverse of $y = 3x + 1$. Then graph $y = 3x + 1$ and its inverse on the same set of axes.

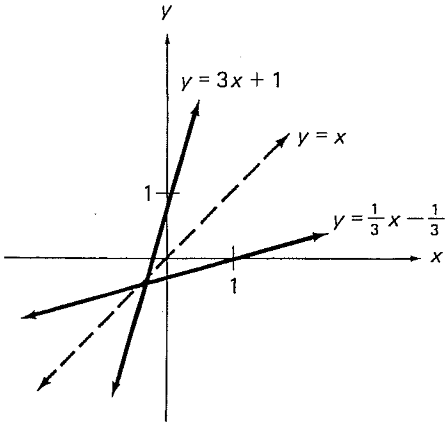


Figure 4-2

Solution

Interchange x and y in $y = 3x + 1$.

$$x = 3y + 1$$

Solve for y in terms of x.

$$y = \frac{1}{3}x - \frac{1}{3}$$

Now graph the function and its inverse (Fig. 4-2).

In Example 1, both the function $y = 3x + 1$ and its inverse are functions. However, as Example 2 illustrates, the inverse of a function may not be a function.

EXAMPLE 2

(a) Graph $y = x^2$ and its inverse.

(b) Write the inverse so that y is given in terms of x.

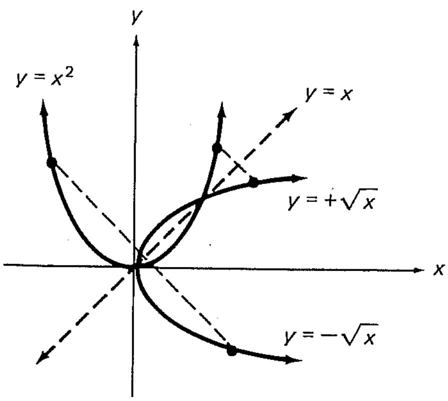


Figure 4-3

Solution

(a) Graph $y = x^2$ and then reflect the graph across the line $y = x$ (Fig. 4-3).

(b) To write the inverse, interchange x and y.

$$x = y^2$$

Solve for y.

$$y = \pm\sqrt{x}$$

The vertical-line test shows that $y = \pm\sqrt{x}$ is not a function.

Figure 4-3 suggests that the domain of a function is the range of its inverse and the range of a function is the domain of its inverse.

Function, $y = x^2$		Inverse, $y = \pm\sqrt{x}$
Domain	Set of real numbers	Range
Range	Set of nonnegative real numbers	Domain

Every function has an inverse, and the trigonometric functions are no exceptions. The inverse of $y = \sin x$ can be written as $x = \sin y$, but new notation is needed to solve this equation for y.

Definition $y = \sin^{-1} x$ means $\sin y = x$.

The equation $y = \sin^{-1} x$, also written $y = \arcsin x$, means "y is a number whose sine is x."

EXAMPLE 3

Use the (SIN) key on your calculator to find, to the nearest hundredth, $y = \sin^{-1} 0.75$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Solution

The equation $y = \sin^{-1} 0.75$ means "y is a number whose sine is 0.75." Thus, the goal is to find a number y such that $\sin y$ is very close to 0.75 and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The graph of the sine curve (Fig. 4-4) shows that a goal number y is between 0 and $\frac{\pi}{2}$, a little closer to $\frac{\pi}{2}$ than to 0. Pick a first estimate, say 1. Switch your calculator to radian mode and enter 1 (SIN). The display reads (.84147099)

$$\sin 1 \approx 0.841 \quad \text{or} \quad \sin^{-1} 0.841 \approx 1$$

Record these values in a table.

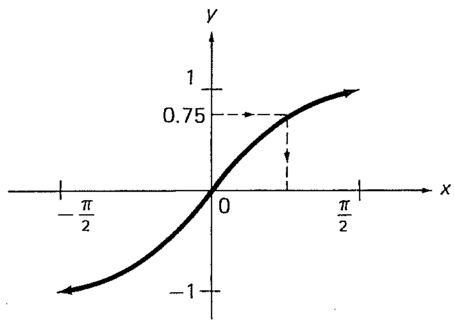


Figure 4-4

x	$y = \sin^{-1} x$
0	0
1	$\frac{\pi}{2}$
0.841	1
0.717	0.8
0.783	0.9
0.751	0.85

Our first estimate was too large, so try a smaller estimate, say 0.8. Enter .8 (SIN).

$$\sin 0.8 \approx 0.717 \quad \text{or} \quad \sin^{-1} 0.717 \approx 0.8$$

Since $0.717 < 0.75 < 0.841$, choose the next estimate between 0.8 and 1, say 0.9. Use the (SIN) key to find that $\sin^{-1} 0.783 \approx 0.9$. Since 0.717

$< 0.75 < 0.783$, choose the next estimate between 0.8 and 0.9. Try 0.85 and find that $\sin^{-1} 0.751 \approx 0.85$. This is very close to 0.75. In fact, $\sin^{-1} 0.745 \approx 0.84$, and 0.751 is much closer to 0.75 than is 0.745.

Answer
To the nearest hundredth, $\sin^{-1} 0.75 = 0.85$.

The correspondence $y = \sin^{-1} x$ is not a function. In fact, there are infinitely many values of y for any x in the domain. For example, the equation $y = \sin^{-1} \frac{1}{2}$ can be interpreted as "y is any number whose sine is $\frac{1}{2}$." Thus, y may be any of the following numbers:

$$\ldots -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$$

This set of numbers can be written as follows:

$$\sin^{-1} \frac{1}{2} = \left\{y\colon y = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad y = \frac{5\pi}{6} + 2n\pi,\ n \text{ an integer}\right\}$$

The inverse of the cosine function can be defined in a manner similar to that of the sine. That is, $\boldsymbol{y = \cos^{-1} x}$ means $\cos y = x$. The equation $y = \cos^{-1} x$, also written $\boldsymbol{y = \arccos x}$, means "$y$ is a number whose cosine is x."

EXAMPLE 4

Find the following in terms of radians:

(a) $\cos^{-1}\left(-\frac{1}{2}\right)$ (b) $\arcsin \frac{\sqrt{2}}{2}$

Solution
(a) If $y = \cos^{-1}\left(-\frac{1}{2}\right)$, then $\cos y = -\frac{1}{2}$. Therefore,

$$y = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad y = \frac{4\pi}{3} + 2n\pi$$

where n is any integer.

(b) If $y = \arcsin \frac{\sqrt{2}}{2}$, then $\sin y = \frac{\sqrt{2}}{2}$. Therefore,

$$y = \frac{\pi}{4} + 2n\pi \quad \text{or} \quad y = \frac{3\pi}{4} + 2n\pi$$

where n is any integer.

Exercises

Set A

In exercises 1 to 8, graph each function and its inverse on the same set of axes. Determine whether each inverse is a function.

1 $y = 2x - 3$ **2** $y = 6x$ **3** $y = 5$

4 $y = -2$ **5** $y = -2x^2$ **6** $y = x^2 + 2$

7 $y = |x|$ **8** $y = x^3$

In exercises 9 to 16, find all values of each expression in terms of radians.

9 $\sin^{-1} 0$ **10** $\cos^{-1} 1$ **11** $\cos^{-1} \frac{1}{2}$

12 $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **13** $\sin^{-1} 1$ **14** $\cos^{-1} 0$

15 $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **16** $\arcsin\left(-\frac{1}{2}\right)$

In exercises 17 to 28, write an equation for the inverse of each function where y is given in terms of x. State the domain and range of the given function and its inverse.

17 $y = 3x + 4$ **18** $y = -2x + 3$

19 $y = 2x + 5$ **20** $y = -3(x + 1)$

21 $y = (x + 1)^2$ **22** $y = 2x^2 - 3$

23 $y = \cos x$ **24** $y = \sin x$

Set B

25 $y = \frac{2}{x}$ **26** $y = \frac{-3}{x + 2}$ **27** $y = 3^x$ **28** $y = 3x^3$

29 State the domain and range of $y = \sqrt{x}$ and graph. Is $y = \sqrt{x}$ a function? Find the inverse of $y = \sqrt{x}$, state its domain and range, and graph. Is the inverse a function?

■ **30** Example 2 showed that $y = x^2$ is a function, but its inverse, $y = \pm\sqrt{x}$, is not. Restrict the domain of $y = x^2$ so that $x \geq 0$ and graph the inverse. Is the inverse a function? What is its equation?

In exercises 31 to 34, graph each function and its inverse on the same set of axes. Determine if the inverse is a function. If not, restrict the domain of the given function to the largest subset of real numbers for which the inverse of the restricted function is a function.

■ **31** $y = x^2 + 1$ ■ **32** $y = 2^x$

■ **33** $y = \sin x$ ■ **34** $y = \cos x$

35 Refer to exercises 1 to 8 and 31 to 34. What property is characteristic of all functions whose inverses are functions?

Use the (SIN) or (COS) key on your calculator to find to the nearest hundredth the values in exercises 36 to 39 in radians. Restrict $y = \sin^{-1} x$ so that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, and restrict $y = \cos^{-1} x$ so that $0 \le y \le \pi$.

■ **36** $\sin^{-1} 0.32$ ■ **37** $\cos^{-1} 0.75$

■ **38** $\cos^{-1} (-0.56)$ ■ **39** $\sin^{-1} (-0.65)$

Set C

40 Prove that a function $y = f(x)$ is one to one if and only if $y = f^{-1}(x)$ is a function.

41 Use the (SIN) key on your calculator to find, to the nearest hundredth, all values of $\sin^{-1} (-0.83)$ in radians.

4-2 INVERSE SINE AND COSINE FUNCTIONS

The graph of $y = \sin^{-1} x$ (Fig. 4-5) is the reflection image of the graph of $y = \sin x$ with respect to the line $y = x$. The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$, and the range is the set of real numbers. There are infinitely many ways to restrict the range of $y = \sin^{-1} x$ to make it a function that still retains the domain $-1 \le x \le 1$. One way is to define y as between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Alternatively, y could be restricted so that

$$\frac{\pi}{2} \le y \le \frac{3\pi}{2} \qquad \text{or} \qquad -\frac{3\pi}{2} \le y \le -\frac{\pi}{2}.$$

For example, $y = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$ if y is restricted so that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,

$$y = \sin^{-1} \frac{1}{2} = \frac{5\pi}{6} \quad \text{if} \quad \frac{\pi}{2} \le y \le \frac{3\pi}{2},$$

and

$$y = \sin^{-1} \frac{1}{2} = -\frac{7\pi}{6} \text{ if } -\frac{3\pi}{2} \le y \le -\frac{\pi}{2}.$$

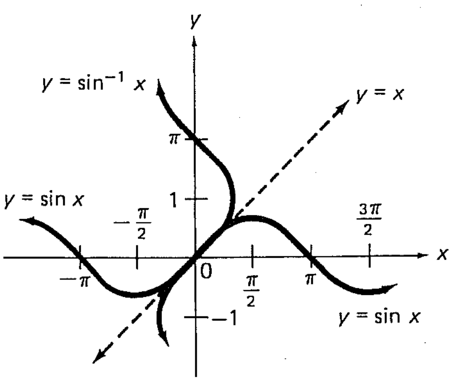


Figure 4-5

Calculators are designed to compute one value of $\sin^{-1} x$ for each x.

EXAMPLE 1

With your calculator set in radian mode, use the (INV) and (SIN) keys (or the (SIN⁻¹) key) to find the following values:

(a) $\sin^{-1} 0.75$ (b) $\arcsin(-0.32)$

Solution

(a) Switch your calculator to radian mode and follow this key sequence:

. 7 5 (INV) (SIN)

(If your calculator has a (SIN⁻¹) key, press it instead of (INV) and (SIN).) The display shows (.84806208), which is a value of $\sin^{-1} 0.75$ to eight decimal places.

(b) Switch your calculator to radian mode and follow this key sequence:

. 3 2 (+/−) (INV) (SIN)

The display shows (−.32572949), which is a value of arcsin (-0.32) to eight decimal places.

Notice that the displayed solutions in Example 1 are between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ $\left(\text{since } \frac{\pi}{2} \approx 1.57\right)$. Calculators (and mathematicians as well) use this restricted range. The numbers in this restricted range are called the **principal values** of the inverse sine. The function with this restriction is called the **inverse sine function** and is written $\boldsymbol{y = \mathrm{Sin}^{-1}\, x}$ or $\boldsymbol{y = \mathrm{Arcsin}\, x}$ (note that capital letters are used when the inverses are functions).

Definition $\boldsymbol{y = \mathrm{Sin}^{-1}\, x}$ means $y = \sin^{-1} x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

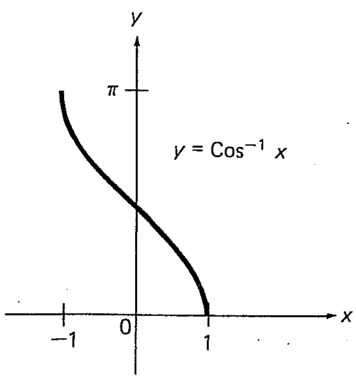


Figure 4-6

In a similar way, the range of $y = \cos^{-1} x$ can be restricted so that $0 \le y \le \pi$. The numbers in this restricted range are called the **principal values** of the inverse cosine. The function with this restriction is called the **inverse cosine function** and is written $\boldsymbol{y = \mathrm{Cos}^{-1}\, x}$ or $\boldsymbol{y = \mathrm{Arccos}\, x}$ (see Fig. 4-6).

Definition $\boldsymbol{y = \mathrm{Cos}^{-1}\, x}$ means $y = \cos^{-1} x$ and $0 \le y \le \pi$.

EXAMPLE 2

Find the exact value of y in radians:

(a) $y = \text{Cos}^{-1}\left(-\frac{1}{2}\right)$ (b) $y = \text{Sin}^{-1}\frac{\sqrt{2}}{2}$

Solution

(a) Since $0 \leq y \leq \pi$ and $\cos y = -\frac{1}{2}$, y must be in quadrant II.

$$y = \text{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

(b) Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin y = \frac{\sqrt{2}}{2}$, y must be in quadrant I.

$$y = \text{Sin}^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

Calculators can also be used to approximate values of the inverse cosine function for any numbers in the domain, as Example 3 shows.

EXAMPLE 3

Use the (INV) and (COS) (or the (COS^{-1})) keys to find the following values, first in radians and then in degrees:

(a) $\text{Cos}^{-1}(-0.12)$ (b) Arccos 0.91

Solution

(a) Switch your calculator to radian mode and use the following key sequence:

. 1 2 (+/−) (INV) (COS)

The displayed value is the approximate solution.

$$\text{Cos}^{-1}(-0.12) \approx 1.6911$$

To find this value in degrees, follow the same key sequence but use degree mode.

$$\text{Cos}^{-1}(-0.12) \approx 96.89°$$

(b) Switch your calculator to radian mode and use the following key sequence:

. 9 1 (INV) (COS)

The displayed value is the approximate solution.

Arccos $0.91 \approx 0.4275$

To find this value in degrees, follow the same key sequence but use degree mode.

Arccos $0.91 \approx 24.49°$

Consider $\sin\left(\mathrm{Sin}^{-1}\frac{1}{2}\right)$. By definition, $\mathrm{Sin}^{-1}\frac{1}{2}$ is the number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$. Hence, $\sin\left(\mathrm{Sin}^{-1}\frac{1}{2}\right) = \frac{1}{2}$. This suggests the following properties, which are direct consequences of the definitions of the inverse functions:

$\mathbf{\sin(Sin^{-1}\,x) = x}$ and $\mathbf{\cos(Cos^{-1}\,x) = x}$

It does *not* follow that $\mathrm{Sin}^{-1}(\sin x)$ is x or that $\mathrm{Cos}^{-1}(\cos x)$ is x unless x is in the domain of that inverse function.

EXAMPLE 4

Use your calculator set in radian mode to evaluate the following:

(a) $\mathrm{Sin}^{-1}(\sin 4.3)$ (b) $\mathrm{Cos}^{-1}(\cos 3)$

Solution

(a) Notice that $4.3 > \frac{\pi}{2}$ and so is not in the domain of $y = \mathrm{Sin}^{-1}x$. Switch your calculator to radian mode and use the following key sequence:

4 . 3 (SIN) (INV) (SIN)

The result is

$\mathrm{Sin}^{-1}(\sin 4.3) \approx -1.1584$

(b) In this case, 3 is in the domain of $y = \mathrm{Cos}^{-1}x$ since $0 \le 3 \le \pi$. Switch your calculator to radian mode and use the following key sequence:

3 (COS) (INV) (COS)

The result is

$\mathrm{Cos}^{-1}(\cos 3) = 3$

Exercises

Set A

In exercises 1 to 4, give the domain and range for each correspondence. Determine if each is a function.

1 $y = \sin^{-1} x$

2 $y = \text{Cos}^{-1} x$

3 $y = \text{Sin}^{-1} x$

4 $y = \cos^{-1} x$

In exercises 5 to 12, find the exact value in radians. Do not use your calculator.

5 $\text{Sin}^{-1} 1$

6 $\text{Cos}^{-1} \dfrac{\sqrt{3}}{2}$

7 $\text{Arcsin}\left(-\dfrac{1}{2}\right)$

8 $\text{Arccos}(-1)$

9 $\text{Sin}^{-1} 0$

10 $\text{Cos}^{-1} \dfrac{1}{2}$

11 $\sin(\text{Sin}^{-1} 0.82)$

12 $\cos[\text{Cos}^{-1}(-0.46)]$

In exercises 13 to 36, use your calculator set in radian mode to evaluate each expression to the nearest hundredth.

13 $\text{Cos}^{-1}(-0.27)$

14 $\text{Sin}^{-1}(-0.91)$

15 $\text{Cos}^{-1} 0.15$

16 $\text{Arccos}(-0.17)$

17 $\text{Arcsin } 0.25$

18 $\text{Sin}^{-1} 0.44$

19 $\text{Sin}^{-1}(-0.84)$

20 $\text{Cos}^{-1} 0.22$

21 $\text{Sin}^{-1} 0.73$

22 $\text{Sin}^{-1}(\sin 1)$

23 $\text{Cos}^{-1}(\cos 1)$

24 $\text{Cos}^{-1}(\cos 2.73)$

25 $\text{Sin}^{-1}[\sin(-3)]$

26 $\text{Sin}^{-1}[\sin(-54)]$

27 $\text{Cos}^{-1}(\cos 73)$

28 $\text{Cos}^{-1}[\cos(-18)]$

Set B

29 $\text{Cos}^{-1} \dfrac{5 + \sqrt{2}}{7}$

30 $\text{Cos}^{-1} \dfrac{7}{2\pi}$

31 $\text{Sin}^{-1}(3 - \sqrt{17})$

32 $\text{Sin}^{-1}(\cos 3.41)$

33 $\text{Sin}^{-1}(\cos 1.38)$

34 $\text{Arccos}(\cot 20)$

35 $\text{Arccos}[\sin(-12)]$

36 $\cos(\text{Arcsin } 0.42)$

In exercises 37 to 42, determine whether the calculator that gave these results was in radian or degree mode. Do not use your calculator.

37 $\text{Sin}^{-1} 0.23 \approx 13.30$

38 $\text{Cos}^{-1}(-0.14) \approx 1.711$

39 $\text{Cos}^{-1}(-0.86) \approx 2.606$ 40 $\text{Sin}^{-1} 0.52 \approx 31.33$

41 $\text{Sin}^{-1}(-0.11) \approx -6.315$ 42 $\text{Cos}^{-1} 0.99 \approx 8.110$

Use an angle in standard position to compute the values in exercises 43 to 46 exactly.

43 $\cos\left(\text{Sin}^{-1}\frac{3}{5}\right)$ 44 $\tan\left(\text{Sin}^{-1}\frac{3}{5}\right)$

45 $\sec\left(\text{Sin}^{-1}\frac{3}{5}\right)$ 46 $\csc\left(\text{Sin}^{-1}\frac{3}{5}\right)$

For exercises 47 to 52 determine if the given equation *may* be an identity by substituting $x = \frac{1}{2}$ and $x = 0.9$.

47 $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$ 48 $\text{Cos}^{-1} x = \frac{\pi}{2} - \text{Sin}^{-1} x$

49 $\text{Cos}^{-1} x = \text{Cos}^{-1}(-x)$ 50 $\text{Sin}^{-1} 2x = 2\,\text{Sin}^{-1} x\,\text{Cos}^{-1} x$

51 $\cos(\text{Sin}^{-1} x) = \sqrt{1 - x^2}$ 52 $\sin(\text{Cos}^{-1} x) = \sqrt{1 - x^2}$

53 Estimate $y = \text{Sin}^{-1} 0.8$ using the graph of $y = \text{Sin}^{-1} x$. Then compute this value with your calculator.

54 Estimate $y = \text{Cos}^{-1} 0.8$ using the graph of $y = \text{Cos}^{-1} x$. Then compute this value with your calculator.

55 Use your calculator to attempt to find $\text{Sin}^{-1} 2$. Explain the result.

56 Use your calculator to attempt to find $\text{Cos}^{-1}(-1.5)$. Explain the result.

■ 57 Graph the inverse of $y = \tan x$ by reflecting the graph of $y = \tan x$ across the line $y = x$. Find the domain and range of this correspondence. How could the domain of $y = \tan x$ be restricted so its inverse is a function whose domain is the range of $y = \tan x$?

■ 58 Graph the inverse of $y = \cot x$ by reflecting the graph of $y = \cot x$ across the line $y = x$. Find the domain and range of this correspondence. How could the domain of $y = \cot x$ be restricted so its inverse is a function whose domain is the range of $y = \cot x$?

Set C

In exercises 59 to 64, verify the given identity where $-1 \le x \le 1$.

59 $\cos(\text{Sin}^{-1} x) = \sqrt{1 - x^2}$ 60 $\sin(\text{Cos}^{-1} x) = \sqrt{1 - x^2}$

61 $\tan(\text{Sin}^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$ 62 $\tan(\text{Cos}^{-1} x) = \dfrac{\sqrt{1 - x^2}}{x}$

63 $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$ 64 $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x$

65 Graph $y = \pi + \text{Sin}^{-1} 2x$. 66 Graph $y = \pi + 2\,\text{Cos}^{-1} x$.

4-3 INVERSE TANGENT AND COTANGENT FUNCTIONS

The inverse of the tangent function is written

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$

The inverse tangent is not a function. For example, $\tan^{-1} 0$ may be 0, π, 2π, or, in general, $k\pi$, where k is any integer. However, if the range is restricted, as in Fig. 4-7, the inverse tangent is a function.

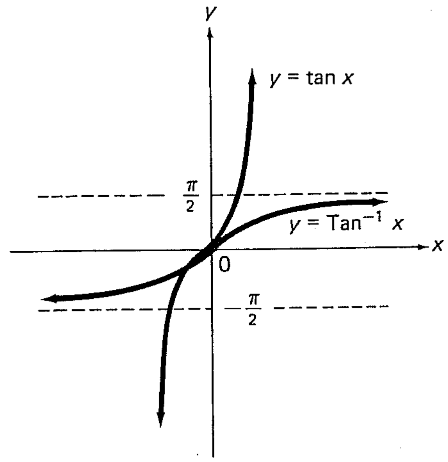


Figure 4-7

Definition $y = \mathrm{Tan}^{-1} x$ means $y = \tan^{-1} x$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

The inverse tangent function is also written y = **Arctan** x and numbers in its range are called **principal values** of the inverse tangent.

EXAMPLE 1

Evaluate the following in radians using the (INV) and (TAN) (or (TAN⁻¹)) keys on your calculator:

(a) $\mathrm{Tan}^{-1} 2$ (b) $\mathrm{Tan}^{-1} (-0.6)$

Solution

(a) Set your calculator in radian mode and use the following key sequence:

2 (INV) (TAN)

To four decimal places,

$$\mathrm{Tan}^{-1} 2 \approx 1.1071$$

(b) Set your calculator in radian mode and use the following key sequence:

. 6 (+/−) (INV) (TAN)

To four decimal places,

$\text{Tan}^{-1}(-0.6) \approx -0.5404$

In a similar manner, the graph of $y = \cot^{-1} x$ can be obtained from $y = \cot x$. As with the other trigonometric functions, the domain of $y = \cot x$ must be restricted for its inverse $y = \cot^{-1} x$ to be a function. This is done in Fig. 4-8.

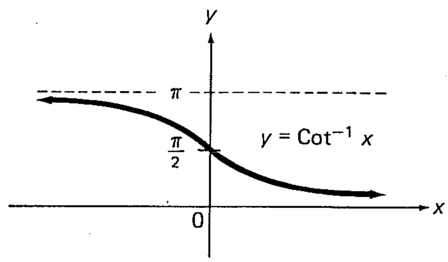


Figure 4-8

Definition $y = \text{Cot}^{-1} x$ means $y = \cot^{-1} x$ and $0 < y < \pi$.

The inverse cotangent function is also written y = **Arccot** $\boldsymbol{x}$ and numbers in its range are called **principal values** of the inverse cotangent.

EXAMPLE 2

Find exact values of the following in terms of radians:

(a) $y = \text{Cot}^{-1} 1$ (b) $y = \text{Arctan}(-\sqrt{3})$

Solution

(a) Since $0 < y < \pi$ and $\cot y = 1$, y must be in quadrant I.

$$y = \text{Cot}^{-1} 1 = \frac{\pi}{4}$$

(b) Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan y = -\sqrt{3}$, y must be in quadrant IV.

$$y = \text{Arctan}(-\sqrt{3}) = -\frac{\pi}{3}$$

Most calculators do not have a $\boxed{\text{COT}}$ or $\boxed{\text{COT}^{-1}}$ key, but the identity $\tan y = 1/\cot y$ can be used to compute values of the inverse cotangent function. The method is illustrated in Example 3.

EXAMPLE 3

Use your calculator to evaluate the following in terms of radians:

(a) $y = \text{Cot}^{-1}\, 0.6513$ (b) $\text{Cot}^{-1}\,(-4.751)$

Solution

(a) If $y = \text{Cot}^{-1}\, 0.6513$, then $\cot y = 0.6513$ and $0 < y < \pi$. Use the identity $\tan y = 1/\cot y$. Then $\tan y = 1/0.6513$ and $y = \text{Tan}^{-1}\,(1/0.6513)$, provided $0 < y < \pi$. We can use the following key sequence to find y:

. 6 5 1 3 $\boxed{\text{1/X}}$ $\boxed{\text{INV}}$ $\boxed{\text{TAN}}$

The result is

$$y = \text{Cot}^{-1}\, 0.6513 \approx 0.9935$$

(b) As in part (a), use the following key sequence in radian mode:

4 . 7 5 1 $\boxed{+/-}$ $\boxed{\text{1/X}}$ $\boxed{\text{INV}}$ $\boxed{\text{TAN}}$

The result is approximately -0.2075, a quadrant IV value of $\cot^{-1}\,(-4.751)$. To find $\text{Cot}^{-1}\,(-4.751)$, which is in quadrant II, add π to -0.2075.

$$\begin{aligned} \text{Cot}^{-1}\,(-4.751) &\approx \pi + (-0.2075) \\ &\approx 2.9341 \end{aligned}$$

In Example 4, angles in standard position are used to evaluate expressions involving inverse trigonometric functions.

EXAMPLE 4

Find the exact values of the following:

(a) $\sin\left(\text{Tan}^{-1}\,\dfrac{4}{3}\right)$ (b) $\cot\left(\text{Sin}^{-1}\,\dfrac{5}{8}\right)$

Solution

(a) $\text{Tan}^{-1}\frac{4}{3}$ can be viewed as an angle θ in standard position. Use the Pythagorean theorem to find $r = 5$. The sine of this angle is y/r. $\sin\left(\text{Tan}^{-1}\frac{4}{3}\right) = \frac{4}{5}$ (Fig. 4-9).

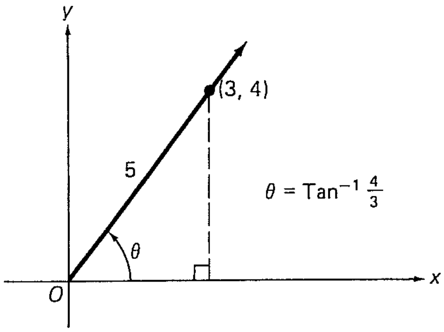


Figure 4-9

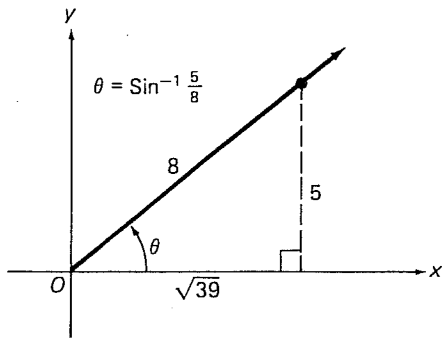


Figure 4-10

(b) Draw an angle θ in standard position as in part (a). Show that $x = \sqrt{39}$. The cotangent of this angle is x/y. $\cot\left(\text{Sin}^{-1}\frac{5}{8}\right) = \frac{\sqrt{39}}{5}$ (Fig. 4-10).

Exercises

Set A

In exercises 1 to 16, find exact values in radians. Do not use your calculator.

1 $\text{Tan}^{-1} 1$ **2** $\text{Tan}^{-1} 0$ **3** $\text{Cot}^{-1} 0$

4 $\text{Cot}^{-1} \sqrt{3}$ **5** $\text{Tan}^{-1} \sqrt{3}$ **6** $\text{Tan}^{-1} \frac{\sqrt{3}}{3}$

7 $\text{Cot}^{-1} \frac{\sqrt{3}}{3}$ **8** $\text{Arccot}\,(-1)$ **9** $\text{Arctan}\,(-1)$

10 $\text{Arctan}\left(-\frac{\sqrt{3}}{3}\right)$ **11** $\text{Arccot}\left(-\frac{\sqrt{3}}{3}\right)$ **12** $\text{Cot}^{-1}(-\sqrt{3})$

13 $\tan(\text{Tan}^{-1} 1)$ **14** $\cot(\text{Cot}^{-1} 0)$ **15** $\tan(\text{Cot}^{-1} \sqrt{3})$

16 $\cot(\text{Tan}^{-1} \sqrt{3})$

In exercises 17 to 34, use your calculator set in radian mode to evaluate each function to the nearest hundredth.

17 $\text{Tan}^{-1} 0.86$ **18** $\text{Tan}^{-1} 10.65$

19 $\text{Tan}^{-1}(-3.96)$

20 $\text{Tan}^{-1}(-8.73)$

21 $\text{Cot}^{-1} 9.62$

22 $\text{Cot}^{-1} 3.78$

23 $\text{Cot}^{-1}(-0.55)$

24 $\text{Cot}^{-1}(-3.91)$

25 $\text{Tan}^{-1} 12.6$

26 $\text{Cot}^{-1}(-6.24)$

27 $\text{Tan}^{-1}(-12.6)$

28 $\text{Cot}^{-1} 6.24$

Set B

29 $\text{Tan}^{-1}(\tan 1.32)$

30 $\text{Tan}^{-1}[\tan(-0.86)]$

31 $\text{Tan}^{-1}(\tan 5.71)$

32 $\text{Tan}^{-1}[\tan(-26.19)]$

33 $\text{Tan}^{-1}(\tan 3.41)$

34 $\text{Tan}^{-1}[\tan(-1.53)]$

35 Refer to exercises 25 and 27. Make a conjecture about the relationship between $\text{Tan}^{-1} x$ and $\text{Tan}^{-1}(-x)$

36 Refer to exercises 26 and 28. Make a conjecture about the relationship between $\text{Cot}^{-1} x$ and $\text{Cot}^{-1}(-x)$.

37 Refer to exercises 29 to 34. Make a conjecture concerning the values of x for which $\text{Tan}^{-1}(\tan x) = x$.

38 Explain why $\tan(\text{Tan}^{-1} x) = x$ and $\cot(\text{Cot}^{-1} x) = x$ for all permissible values of x.

In exercises 39 to 50, find exact values by referring to an angle in standard position.

39 $\cos\left(\text{Tan}^{-1}\frac{4}{3}\right)$

40 $\tan\left(\text{Cos}^{-1}\frac{3}{5}\right)$

41 $\sin(\text{Cot}^{-1} 3)$

42 $\cos\left(\text{Tan}^{-1}\frac{5}{12}\right)$

43 $\tan\left(\text{Sin}^{-1}\frac{2}{3}\right)$

44 $\cos(\text{Cot}^{-1} 5)$

45 $\sin\left[\text{Tan}^{-1}\left(-\frac{5}{3}\right)\right]$

46 $\sin\left[\text{Cot}^{-1}\left(-\frac{7}{4}\right)\right]$

47 $\sec\left(\text{Cot}^{-1}\frac{10}{7}\right)$

48 $\csc\left(\text{Tan}^{-1}\frac{5}{8}\right)$

49 $\csc[\text{Tan}^{-1}(-2)]$

50 $\sec[\text{Cot}^{-1}(-3)]$

■ **51** Graph the inverse of $y = \sec x$ by reflecting the graph of $y = \sec x$ across the line $y = x$. Find the domain and range of this correspondence. How could the domain of $y = \sec x$ be restricted so its inverse is a function whose domain is the range of $y = \sec x$?

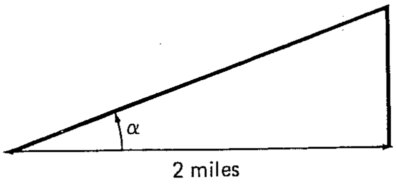


Figure 4-11

■ **52** Graph the inverse of $y = \csc x$ by reflecting the graph of $y = \csc x$ across the line $y = x$. Find the domain and range of this correspondence. How could the domain of $y = \csc x$ be restricted so its inverse is a function whose domain is the range of $y = \csc x$?

53 The railroad track in Fig. 4-11 must rise 800 ft over a 2-mi horizontal distance. If this is done at a constant grade α, find α in degrees.

54 The clay duck in Fig. 4-12 is projected upward to a height of 50 m. A marksman standing 100 m away wants to shoot the duck at its peak. Find α if he holds his gun 2 m above the ground.

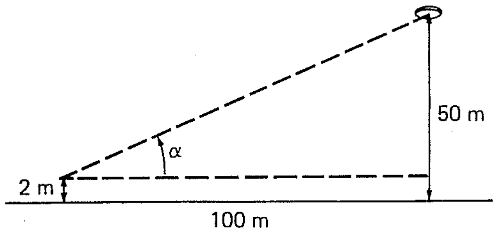


Figure 4-12

Set C

Verify the identities in exercises 55 to 58.

55 $\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1} x$ **56** $\mathrm{Cot}^{-1}(-x) = \pi - \mathrm{Cot}^{-1} x$

57 $\tan(\mathrm{Tan}^{-1} x - \mathrm{Tan}^{-1} y) = \dfrac{x - y}{1 + xy}$

58 $\mathrm{Cos}^{-1} x = \dfrac{\pi}{2} - \mathrm{Tan}^{-1} \dfrac{x}{\sqrt{1 - x^2}}$

59 Graph $y = \pi + 2\,\mathrm{Tan}^{-1} x$ **60** Graph $y = \pi + \mathrm{Cot}^{-1} 2x$

Algebra Review

EXAMPLE 1

Factor $2x^2 - 11x + 12$.

Solution

Use the general equation

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

If $(ax + b)(cx + d)$ is a factorization of the given polynomial, then $ac = 2$ and $bd = 12$. Furthermore, a, b, c, and d must be chosen so that $ad + bc = -11$. By systematic trial and error we find that

$$2x^2 - 11x + 12 = (2x - 3)(x - 4)$$

EXAMPLE 2

Solve $2x^2 - 5x = 6x - 12$.

Solution

Get all terms to the left side.

$$2x^2 - 11x + 12 = 0$$

Factor (see Example 1).

$$(2x - 3)(x - 4) = 0$$

Set each factor equal to 0 and solve.

$$2x - 3 = 0 \qquad x - 4 = 0$$

$$x = \frac{3}{2} \qquad x = 4$$

The solutions are $\frac{3}{2}$ and 4.

In exercises 1 to 6, factor each expression.

1 $y^2 - 7y + 10$ **2** $y^2 + 8y - 9$

3 $2x^2 - 7x + 6$ **4** $2x^2 - 5x - 3$

5 $3u^2 - 7u + 2$ **6** $3u^2 + 21u + 30$

In exercises 7 to 12, solve each equation.

7 $x^2 - 10x + 25 = 0$ **8** $x^2 + 6x + 9 = 0$

9 $2x^2 = x + 3$ **10** $2x^2 + 6x = 8$

11 $6y^2 = 11y - 3$ **12** $6y^2 + 5 = 17y$

4-4

INVERSE SECANT AND COSECANT FUNCTIONS

The inverse of the secant function is written $y = \sec^{-1} x$ or $y = \text{arcsec}\, x$, and the inverse of the cosecant function is written $y = \csc^{-1} x$ or $y = \text{arccsc}\, x$. Neither inverse is a function, but with proper restrictions we can define the **inverse secant function $y = \text{Sec}^{-1} x$** and the **inverse cosecant function $y = \text{Csc}^{-1} x$**. The graphs of these functions are shown in Figs. 4-13 and 4-14. Notice that the domain in each case consists of those values of x satisfying $|x| \geq 1$.

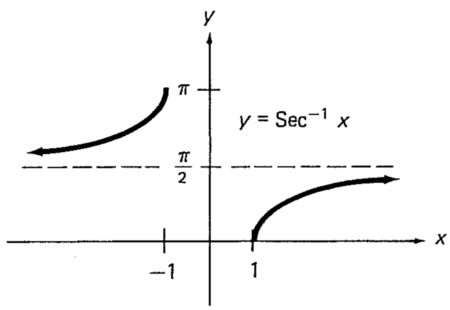


Figure 4-13

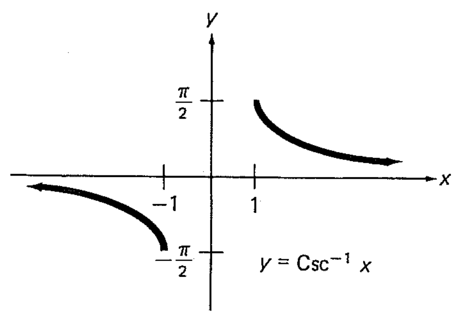


Figure 4-14

Definition

$$y = \text{Sec}^{-1} x \text{ means } y = \sec^{-1} x \text{ and } 0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}.$$

$$y = \text{Csc}^{-1} x \text{ means } y = \csc^{-1} x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0.$$

EXAMPLE 1

Find exact values of the following in terms of radians:

(a) $y = \text{Sec}^{-1} 2$ (b) $y = \text{Csc}^{-1}(-\sqrt{2})$

Solution

(a) Since $0 \leq y \leq \pi$ and $\sec y = 2$, y must be in quadrant I.

$$y = \text{Sec}^{-1} 2 = \frac{\pi}{3}$$

(b) Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\csc y = -\sqrt{2}$, y must be in quadrant IV.

$$y = \text{Csc}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

Since most calculators do not have (SEC), (CSC), (SEC⁻¹), or (CSC⁻¹) keys, the identities $\cos y = 1/\sec y$ and $\sin y = 1/\csc y$ must be used to compute the respective values of the inverse secant and inverse cosecant functions. This method is illustrated in Example 2.

EXAMPLE 2

Use your calculator to evaluate the following in terms of radians:

(a) $y = \mathrm{Sec}^{-1}(-2.52)$ (b) $y = \mathrm{Csc}^{-1} 7.63$

Solution

(a) If $y = \mathrm{Sec}^{-1}(-2.52)$ then

$$\sec y = -2.52 \quad \text{and} \quad 0 \leq y \leq \pi$$

Use the identity $\cos y = 1/\sec y$.

$$\cos y = -\frac{1}{2.52}$$

So $y = \mathrm{Cos}^{-1}(-1/2.52)$, and we can use the following key sequence in radian mode:

2 . 5 2 (+/−) (1/X) (INV) (COS)

Answer

$y = \mathrm{Sec}^{-1}(-2.52) \approx 1.98$

(b) If $y = \mathrm{Csc}^{-1} 7.63$, then

$$\csc y = 7.63 \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Use the identity $\sin y = 1/\csc y$.

$$\sin y = \frac{1}{7.63}$$

So $y = \mathrm{Sin}^{-1}(1/7.63)$, and we can use the following key sequence in radian mode:

7 . 6 3 (1/X) (INV) (SIN)

Answer

$y = \mathrm{Csc}^{-1} 7.63 \approx 0.13$

Exercises

Set A

In exercises 1 to 16, find exact values in radians. Do not use your calculator.

1 $\mathrm{Sec}^{-1}\ 1$

2 $\mathrm{Csc}^{-1}\ 1$

3 $\mathrm{Csc}^{-1}\ (-1)$

4 $\mathrm{Sec}^{-1}\ (-1)$

5 $\mathrm{Sec}^{-1}\ (-2)$

6 $\mathrm{Csc}^{-1}\ 2$

7 $\mathrm{Csc}^{-1}\ 2\sqrt{3}/3$

8 $\mathrm{Sec}^{-1}\ (-2\sqrt{3}/3)$

9 $\mathrm{Sec}^{-1}\ (-\sqrt{2})$

10 $\mathrm{Csc}^{-1}\ \sqrt{2}$

11 $\mathrm{Arccsc}\ 2\sqrt{3}/3$

12 $\mathrm{Arcsec}\ (-2\sqrt{3}/3)$

13 $\sec\ (\mathrm{Sec}^{-1}\ 1)$

14 $\csc\ (\mathrm{Csc}^{-1}\ 1)$

15 $\csc\ (\mathrm{Sec}^{-1}\ 2\sqrt{3}/3)$

16 $\sec\ [\mathrm{Csc}^{-1}\ (-\sqrt{2})]$

Use your calculator set in radian mode to evaluate exercises 17 to 36 to the nearest hundredth.

17 $\mathrm{Sec}^{-1}\ 1.46$

18 $\mathrm{Sec}^{-1}\ 5.77$

19 $\mathrm{Csc}^{-1}\ 4.08$

20 $\mathrm{Csc}^{-1}\ 1.19$

21 $\mathrm{Sec}^{-1}\ (-8.73)$

22 $\mathrm{Sec}^{-1}\ (-4.72)$

23 $\mathrm{Csc}^{-1}\ (-15.91)$

24 $\mathrm{Csc}^{-1}\ (-10.09)$

Set B

25 $\mathrm{Sec}^{-1}\ (\sec 2.53)$

26 $\mathrm{Sec}^{-1}\ (\sec 0.49)$

27 $\mathrm{Sec}^{-1}\ [\sec\ (-3.07)]$

28 $\mathrm{Sec}^{-1}\ (\sec 5.86)$

29 $\mathrm{Sec}^{-1}\ (\sec 0.06)$

30 $\mathrm{Sec}^{-1}\ (\sec 3.21)$

31 $\mathrm{Csc}^{-1}\ (\csc 0.66)$

32 $\mathrm{Csc}^{-1}\ [\csc\ (-1.44)]$

33 $\mathrm{Csc}^{-1}\ [\csc\ (-3.65)]$

34 $\mathrm{Csc}^{-1}\ (\csc 10.48)$

35 $\mathrm{Csc}^{-1}\ [\csc\ (-1.68)]$

36 $\mathrm{Csc}^{-1}\ (\csc 1.53)$

37 Refer to exercises 25 to 30. Make a conjecture concerning the values of x for which $\mathrm{Sec}^{-1}\ (\sec x) = x$.

38 Refer to exercises 31 to 36. Make a conjecture concerning the values of x for which $\mathrm{Csc}^{-1}\ (\csc x) = x$.

39 Explain why $\sec\ (\mathrm{Sec}^{-1}\ x) = x$ for all x such that $|x| \geq 1$.

40 Explain why $\csc\ (\mathrm{Csc}^{-1}\ x) = x$ for all x such that $|x| \geq 1$.

In exercises 41 to 49, find exact values by referring to an angle in standard position.

41 $\sin(\mathrm{Sec}^{-1} 5/4)$

42 $\sin(\mathrm{Csc}^{-1} 5/4)$

43 $\sec(\mathrm{Csc}^{-1} 5/3)$

44 $\csc(\mathrm{Sec}^{-1} 5/3)$

45 $\tan(\mathrm{Sec}^{-1} 13/5)$

46 $\cot(\mathrm{Csc}^{-1} 13/12)$

47 $\cos(\mathrm{Csc}^{-1} 8/5)$

48 $\cos(\mathrm{Sec}^{-1} 10/7)$

49 $\sin[\mathrm{Sec}^{-1}(-9/5)]$

50 Use your calculator to attempt to find $\mathrm{Csc}^{-1}(-0.9)$. Explain the result.

51 Use your calculator to attempt to find $\mathrm{Sec}^{-1} 0.5$. Explain the result.

■ **52** Solve for x: $2\sin x - 1 = 0$. *Hint*: Solve for $\sin x$ and find the inverse sine of the result.

■ **53** Solve for x: $2\sec x + 3 = 0$.

Set C

Verify the identities in exercises 54 and 55.

54 $\mathrm{Sec}^{-1} x = \pi - \mathrm{Sec}^{-1}(-x)$ if $|x| \geq 1$.

55 $\mathrm{Csc}^{-1}(-x) = -\mathrm{Csc}^{-1} x$ if $|x| \geq 1$.

56 Graph $y = \pi + 2\,\mathrm{Csc}^{-1} x$.

57 Graph $y = 3\,\mathrm{Sec}^{-1} 2x$.

Midchapter Review

Section 4-1

1 Find the inverse of $y = -x + 2$. Graph $y = -x + 2$ and its inverse on the same set of axes. Is the inverse a function?

2 Find the inverse of $y = x^2 - 5$. State the domain and range of $y = x^2 - 5$ and its inverse. Is the inverse a function?

3 Find all values in radians of $\cos^{-1}\sqrt{3}/2$.

Section 4-2

4 Find the exact values in radians of $\mathrm{Sin}^{-1}(-1/2)$ and $\mathrm{Cos}^{-1}(-\sqrt{2}/2)$.

5 Use your calculator set in radian mode to evaluate $\mathrm{Cos}^{-1}(-0.96)$ and $\mathrm{Sin}^{-1} 0.85$ to the nearest hundredth.

6 Explain why $\sin(\mathrm{Sin}^{-1} x) = x$ for all real numbers x.

Section 4-3

7 What are the domain and range of $y = \mathrm{Tan}^{-1} x$? Of $y = \mathrm{Cot}^{-1} x$?

8 Find the exact value in radians of $\tan(\mathrm{Cot}^{-1} \sqrt{3}/3)$.

9 Use your calculator set in radian mode to find $\mathrm{Cot}^{-1} 5.6148$ to four decimal places.

Section 4-4

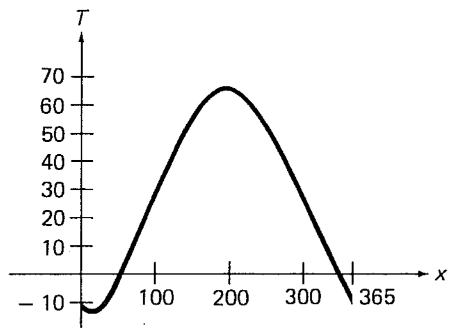
10 What are the domain and range of $y = \mathrm{Sec}^{-1} x$? Of $y = \mathrm{Csc}^{-1} x$?

11 Find the exact value in radians of $\csc(\mathrm{Sec}^{-1} 2)$.

12 Use your calculator set in radian mode to find $\mathrm{Csc}^{-1}(-21.836)$ to the nearest thousandth.

4-5 TRIGONOMETRIC EQUATIONS

In Chapter 3 you saw that the sine function could be used to describe variations in temperature for a given locale. The mean daily Fahrenheit temperature in Fairbanks, Alaska, is graphed in Fig. 4-15. Given a day of the year, the temperature T can be directly computed from the equation. But on what days is T some specified value, say, $T = 0°$? The graph gives an estimate, but we will determine the answer more exactly in Example 3.



$$T = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

Figure 4-15

Example 1 introduces several techniques for solving **conditional trigonometric equations,** that is, equations that are true for some but not all allowable replacements of the variable.

EXAMPLE 1

Solve the following for x:

(a) $2 \sin x + 1 = 0$ (b) $\cos^2 x + 2 \cos x + 1 = 0$

Solution

(a) First solve for the trigonometric term.

$$2 \sin x = -1$$

$$\sin x = -\frac{1}{2}$$

Indicate all values of x that satisfy this equation.

Answer

$x = \frac{7\pi}{6} + 2k\pi$ or $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

(b) The given equation is quadratic where $\cos x$ is the variable. Factor and solve for $\cos x$.

$$(\cos x + 1)^2 = 0$$
$$\cos x + 1 = 0$$
$$\cos x = -1$$

Answer

$x = (2k + 1)\pi$, where k is any integer.

When solving trigonometric equations, it is sometimes helpful to first apply appropriate identities. Example 2 illustrates this technique.

EXAMPLE 2

Solve the following for x:

(a) $\cos 2x + \cos x = 0$ (b) $2 \sin x = \sin 2x$

Solution

(a) Use the identity $\cos 2x = 2 \cos^2 x - 1$.

$$(2 \cos^2 x - 1) + \cos x = 0$$

$$2 \cos^2 x + \cos x - 1 = 0$$

This equation is quadratic in $\cos x$. Factor and solve.

$$(2 \cos x - 1)(\cos x + 1) = 0$$

$$2 \cos x - 1 = 0 \qquad \cos x + 1 = 0$$

$$\cos x = \frac{1}{2} \qquad \cos x = -1$$

Answer

$x = \pm\frac{\pi}{3} + 2k\pi$ or $(2k + 1)\pi$, where k is any integer.

(b) Use the identity $\sin 2x = 2 \sin x \cos x$.

$2 \sin x = 2 \sin x \cos x$

$2 \sin x - 2 \sin x \cos x = 0$

Factor and solve.

$2 \sin x(1 - \cos x) = 0$

$2 \sin x = 0 \qquad 1 - \cos x = 0$

$\sin x = 0 \qquad \cos x = 1$

Answer

$x = k\pi$, where k is any integer.

Not all trigonometric equations have exact solutions. Sometimes it may be necessary to use a scientific calculator or graphing utility to find approximate solutions. The following example illustrates a solution using a scientific calculator. A partial solution using a graphing calculator is given in Example 2 on page 447.

EXAMPLE 3

Refer to Fig. 4-15. On what days of the year is the mean temperature 0°F in Fairbanks?

Solution

Replace T by 0 in the equation given in Fig. 4-15.

$$0 = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

Isolate the sine term.

$$37 \sin\left[\frac{2\pi}{365}(x - 101)\right] = -25$$

$$\sin\left[\frac{2\pi}{365}(x - 101)\right] = -\frac{25}{37} \approx -0.6757$$

Use your calculator's $\boxed{\text{INV}}$ and $\boxed{\text{SIN}}$ keys (in radian mode).

$$\frac{2\pi}{365}(x - 101) \approx -0.7419 + 2k\pi \quad \text{or} \quad -2.3997 + 2k\pi$$

where k is any integer.

Since $1 \leq x \leq 365$, only the two equations which result when $k = 0$ in the first solution and when $k = 1$ in the second are appropriate.

$$\frac{2\pi}{365}(x - 101) \approx -0.7419 \qquad \frac{2\pi}{365}(x - 101) \approx -2.3997 + 2\pi$$

$$2\pi(x - 101) \approx -270.7935 \qquad 2\pi(x - 101) \approx 1417.4721$$

$$x - 101 \approx -43.0981 \qquad x - 101 \approx 225.5977$$

$$x \approx 58 \qquad x \approx 327$$

Answer

The mean temperature in Fairbanks is approximately 0°F on the 58th and on the 327th days of the year, that is, on Feb. 27 and Nov. 23, respectively.

Exercises

Set A

In exercises 1 to 12, find all exact solutions to each equation.

1 $2 \sin x + \sqrt{3} = 0$

2 $2 \cos x = 1$

3 $\cot x - \sqrt{3} = 0$

4 $\sqrt{3} \tan x + 1 = 0$

5 $\sqrt{3} \sec x + 2 = 0$

6 $2 \csc x - 3 = 1$

7 $4 \sin^2 x - 1 = 0$

8 $2 \cos^2 x = 1$

9 $2 \cos^2 x - 5 \cos x + 2 = 0$

10 $4 \sin^2 x + 3 = 0$

11 $3 \sec^2 x - 4 = 0$

12 $\sqrt{3} \csc^2 x + \csc x = 0$

In exercises 13 to 18, use a scientific calculator or a graphing utility to find solutions to four decimal places on the interval $0 \leq x < 2\pi$.

13 $3 \cos x + 2 = 0$

14 $1 - 4 \sin x = 2$

15 $2 \tan^2 x - \tan x = 0$

16 $\cot^2 x + 2 \cot x = 0$

17 $5 \sin (x + 2) = 3$

18 $4 - 5 \cos (2x - 1) = 0$

In exercises 19 to 24, find exact solutions to each equation on the interval $0 \leq x < 2\pi$.

19 $\sin x + \sin 2x = 0$

20 $\sin 2x = \cos x$

21 $\sin 2x + \cos x = 0$

22 $\cos 2x + \sin x = 0$

23 $\cos 2x + 3 \sin x = 2$

24 $\cos 2x - \cos x = 0$

Set B

In exercises 25 to 36, use a scientific calculator or a graphing utility to find solutions to four decimal places on the interval $0 \leq x < 2\pi$.

25 $4 \cos 2x + 3 \cos x = 1$

26 $\sin 2x + 1/2 = \sin x + \cos x$

27 $\sin 2x - 4 \cos 2x = 3$

28 $\tan x + \tan 2x = 0$

29 $\sin 4x - \cos 2x = 0$

30 $\sin 2x - \sin 4x = 0$

31 $\tan^2 x - 5 \tan x = -6$

32 $4 \tan x - \sec^2 x = 0$

33 $\sin 2x \sin x + \cos x = 0$

34 $\sin 2x \cos x - \cos 2x \sin x = 0$

35 $\sin x \sin x/2 + \cos x = 1$

36 $\sin x - \sin x/2 = 0$

37 See Example 3. On what days of the year is the mean temperature 50°F in Fairbanks, AK?

38 See Example 3. On what days of the year is the mean temperature 40°F in Fairbanks?

39 Because of ocean tides, the depth y in meters of the River Thames at London varies as a sine function of x, the hour of the day. On a certain day that function was

$$y = 3 \sin \left[\frac{\pi}{6}(x - 4)\right] + 8$$

where $x = 0, 1, 2, \ldots, 24$ corresponds to 12:00 midnight, 1:00 a.m., 2:00 a.m., . . . , 12:00 midnight the next night. What is the maximum depth of the River Thames on that day? At what time(s) does it occur?

40 What is the minimum depth of the River Thames on the day in exercise 39? At what time(s) does it occur?

41 Find the depth of the River Thames at 12:00 noon on the day in exercise 39. At approximately what time(s) is the depth 10 m?

42 Over what time interval(s) on the day in exercise 39 will the depth of the River Thames be greater than 9 m?

43 The power of an ac circuit is given by the formula $P = EI \cos \theta$, where θ is the phase angle between the voltage and the current. Solve this equation for the phase angle θ.

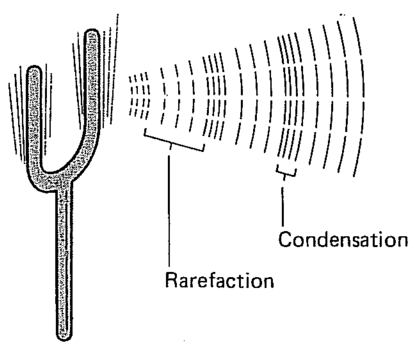


Figure 4-16

44 The formula relating displacement y of a typical air molecule by a simple sound (such as that made by the tuning fork in Fig. 4-16) at time t in seconds is

$$y = D \sin 2\pi ft$$

Here D is the amplitude, or maximum displacement, and f is the frequency

or number of oscillations per second. Find the formula for displacement of an air molecule by a simple sound whose frequency is 300 oscillations per second and whose amplitude is 0.0005 in. Graph one period of this equation.

45 In exercise 44, find the smallest positive value of t for which the displacement is 0.

46 Find the first time (value of t) for which the displacement in exercise 44 is 0.0005 in.

47 Find the first time for which the displacement in exercise 44 is 0.0002 in.

In exercises 48 to 53, use a scientific calculator or a graphing utility to find, to four decimal places, solutions to each equation on the interval $0 \leq x < 2\pi$.

48 $10 \sin (3x + 1) + 15 = 12$

49 $-8 \cos (2x - 5) + 8 = 13$

Set C

50 $\sin x - 2 \cos x = 0$

51 $3 \sin x + \cos x = 0$

52 $\sin 2x + 3 \cos 2x = 0$

53 $\tan^2 2x - 4 \tan 2x = 0$

In exercises 54 to 56, find exact solutions to each equation.

54 $3 \operatorname{Sin}^{-1} x = \frac{1}{2}\pi$

55 $\operatorname{Tan}^{-1} (x - 1) = \frac{\pi}{3}$

56 $\operatorname{Sin}^{-1} (2x - x^2) = \operatorname{Sin}^{-1} \frac{1}{2}$

57 Show that if $\sin x = \sin y$ and $\cos x = \cos y$, then $x = y + 2k\pi$, where k is any integer.

58 Show that if $\sin x = \cos y$ and $\cos x = \sin y$, then $x + y = \frac{1}{2}\pi + 2k\pi$, where k is any integer.

EXTENSION

Equations Involving Trigonometric and Algebraic Terms

Section 4-5 discussed equations involving trigonometric terms and constant terms. If in addition to such terms an equation also contains algebraic terms, the solution becomes more difficult to find. An approach to solving such an equation is illustrated next.

EXAMPLE

Use successive estimation to find x between 0 and $\frac{\pi}{2}$ to the nearest thousandth so that $\cos x = x$.

Solution

Graph $y = \cos x$ and $y = x$ on the same set of axes (see Fig. 4-17). The x coordinate a of the point of intersection of these graphs is the desired solution. It appears to be about 0.75. Test this estimate. Use your calculator set in radian mode to compute $\cos 0.75 - 0.75 \approx -0.0183$. From the graph, we see that if $\cos x - x > 0$, then $a > x$, and if $\cos x - x < 0$, then $a < x$. Since the result for $x = 0.75$ is negative, $a < 0.75$. Try a smaller estimate, say 0.74. The following table shows a series of estimates:

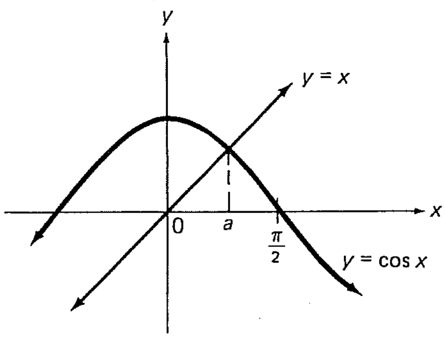


Figure 4-17

Estimated x	0.75	0.74	0.73	0.735	0.739	0.7395
$\cos x - x$	-0.0183	-0.0015	0.0152	-0.0068	0.0001	-0.0007

Answer

Since $0.739 < a$ and $0.7395 > a$, we conclude that to the nearest thousandth the solution of $\cos x = x$ is 0.739.

Exercises

In exercises 1 to 6, apply the method used in the example to find, to the nearest thousandth, the smallest positive solution of each equation. How many solutions does each equation have?

1 $\sin x = x^2$ **2** $\cos x = x^2$ **3** $\sin x = \frac{1}{2}x$

4 $\tan x = 2x$ **5** $x \sin x = 1$ **6** $x \cos x = 1$

7 Use a graphing utility to solve exercises 1–6.

8 The equation $\cos x = x$ was solved in the example. An alternate way to solve this equation is: (a) Set your calculator in radian mode. (b) Enter any number. (c) Press COS. (d) Press COS again. (e) Continue pressing COS until the first four digits in the display stop changing. (f) Your display to the nearest thousandth should be 0.739, the approximate solution of $\cos x = x$. Continue to press COS repeatedly. Each display is a better estimate of the solution. Find the solution to six decimal places.

9 Examine the graph in the example to analyze why the method of exercise 8 works.

10 Try the method of exercise 8 for exercise 1. Use $\sqrt{\sin x} = x$.

4-6

POLAR COORDINATES

Any point in a plane can be specified by an ordered pair of real numbers (x, y). This system of specifying points is called the **rectangular** or Cartesian coordinate system. However, in some cases it is more useful to specify points using a **polar coordinate system.** In such a system, a point O, called the **pole,** and a ray $\overrightarrow{OA}$, called the **polar axis,** are specified. In Fig. 4-18, a severe thunderstorm is spotted on a weather station's (point O) radar screen. The storm's center is at point T. In this polar coordinate system, coordinates of T are $(r, \theta) = (34, 75°)$, indicating that $OT = 34$ km and $\angle AOT = 75°$.

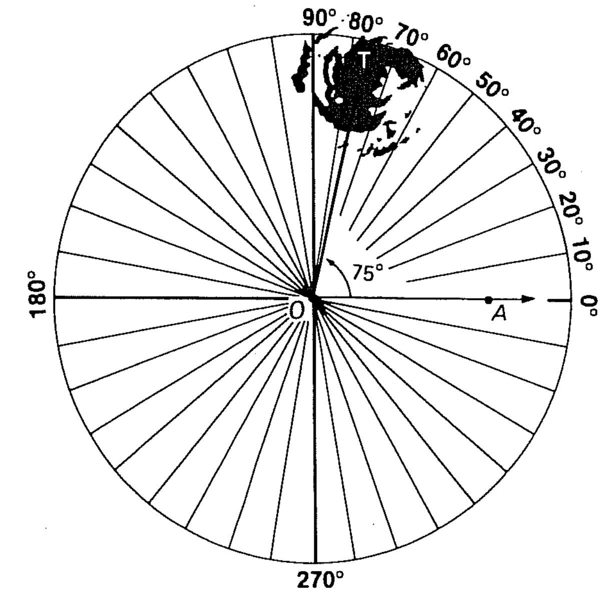


Figure 4-18

In a polar coordinate system, a given point may be specified in many different ways, as Example 1 illustrates.

EXAMPLE 1

Graph points P and Q. Then find two other polar coordinate pairs that represent each point.

(a) $P(5, -170°)$ (b) $Q\left(2, \frac{5\pi}{6}\right)$

Solution

(a) Draw an angle of $-170°$ in standard position. Locate P on the terminal side of this angle so that $OP = 5$ (Fig. 4-19).

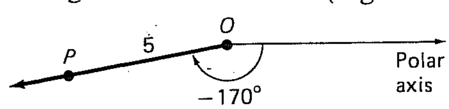


Figure 4-19

Other polar coordinates of P are $(5, 190°)$ and $(5, 450°)$. In general, the value of θ may differ from $-170°$ by a multiple of $360°$.

(b) Draw an angle of $\frac{5\pi}{6}$ in standard position. Locate Q on the terminal side of this angle so that $OQ = 2$ (Fig. 4-20).

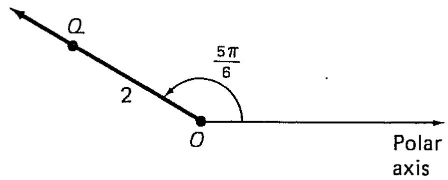


Figure 4-20

Other polar coordinates of Q are $\left(2, \frac{17\pi}{6}\right)$ and $\left(2, \frac{-7\pi}{6}\right)$. In general, the value of θ may differ from $\frac{5\pi}{6}$ by a multiple of 2π.

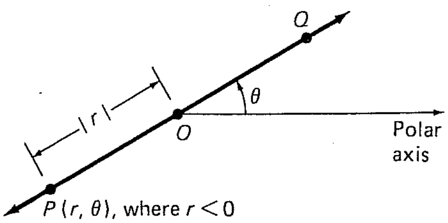


Figure 4-21

Example 1 suggests the following properties for polar coordinates where k is any integer:

$$P(r, \theta) = P(r, \theta + k \cdot 360^\circ) \quad \text{or} \quad P(r, \theta + 2k\pi)$$

The first coordinate r may also be a negative number. In Fig. 4-21, $\overrightarrow{OQ}$ is the terminal side of θ in standard position. Point P is on the ray opposite $\overrightarrow{OQ}$ and $OP = |r|$.

EXAMPLE 2

Graph each of the following points:

(a) $P(-3, 62^\circ)$ (b) $Q\left(-2, \frac{-3\pi}{4}\right)$

Solution

(a) Draw an angle of 62° in standard position. On the ray opposite its terminal side, locate P so that $OP = |-3| = 3$ (Fig. 4-22).

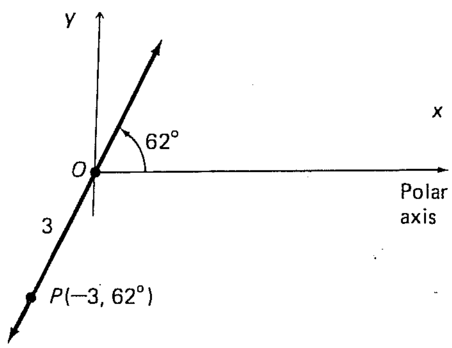


Figure 4-22

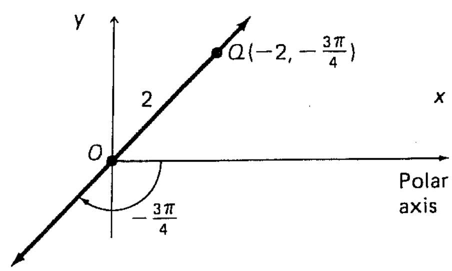


Figure 4-23

(b) Draw an angle of $-\frac{3\pi}{4}$ in standard position. On the ray opposite its terminal side, locate Q so that $OQ = |-2| = 2$ (Fig. 4-23).

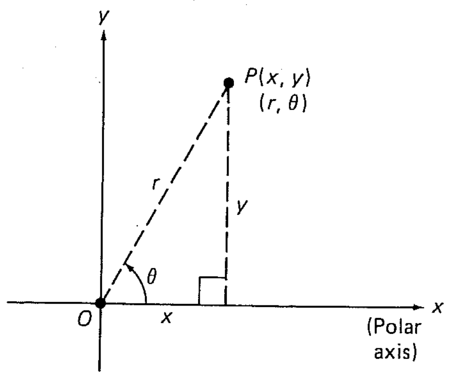


Figure 4-24

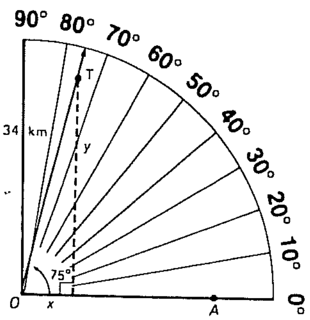
Figure 4-24 shows a rectangular coordinate system. Suppose its origin is the pole and its positive x axis is the polar axis of a polar coordinate system. Then the coordinates of a point P in the two systems are related by the following two equations:

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta$$

These equations can be used to express polar coordinates as rectangular coordinates.

EXAMPLE 3

In Fig. 4-18, $\overrightarrow{OA}$ points east. How far east and how far north is the center of the storm from the weather station?



The x and y coordinates of T in Fig. 4-25 are the desired distance east and distance north, respectively. Use $r = 34$ and $\theta = 75°$ in the equations for x and y:

$$x = r \cos \theta = 34 \cos 75° \approx 8.8$$

$$y = r \sin \theta = 34 \sin 75° \approx 32.8$$

The center of the thunderstorm is located approximately 8.8 km east and 32.8 km north of the weather station.

It is also possible to express polar coordinates of points given their rectangular coordinates and to write equations in polar form which are given in rectangular form and vice versa.

EXAMPLE 4

(a) Write $x^2 + y^2 = 5$ in polar form.

(b) Find a pair of polar coordinates for $P(2, -3)$.

Solution

(a) Substitute $r \cos \theta$ for x and $r \sin \theta$ for y, and simplify.

$$(r \cos \theta)^2 + (r \sin \theta)^2 = 5$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 5$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 5$$

Since $\cos^2\theta + \sin^2\theta = 1$, $r^2 = 5$.

Answer
The rectangular equation $x^2 + y^2 = 5$ is equivalent to the polar equation $r^2 = 5$.

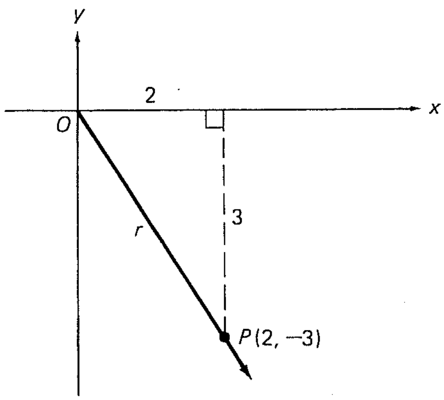


Figure 4-26

(b) Figure 4-26 suggests $r^2 = 2^2 + (-3)^2 = 13$ or $r = \sqrt{13}$

The angle θ can be found using $x = r\cos\theta$.

$$2 = \sqrt{13}\cos\theta \quad \text{or} \quad \cos\theta = \frac{2}{\sqrt{13}}$$

Since P is in quadrant IV, $\theta \approx -56.3°$.

Answer
One pair of polar coordinates of P is $(\sqrt{13}, -56.3°)$.

Example 4 suggests an important relationship between the polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point:

$$\mathbf{r^2 = x^2 + y^2}$$

Exercises

Set A
In exercises 1 to 8, the polar coordinates of a point P are given. Graph P and find a pair (r, θ) for P where $r > 0$ and $0° \leq \theta < 360°$.

1 $(2, -196°)$ **2** $(3, -250°)$ **3** $(5, -76°)$

4 $(8, -31°)$ **5** $(-3, 56°)$ **6** $(-4, 99°)$

7 $(-7, -420°)$ **8** $(-1, -530°)$

In exercises 9 to 16, find the rectangular coordinates, to the nearest tenth, of the points with the following polar coordinates.

9 $\left(4, \frac{\pi}{2}\right)$ **10** $(7, -\pi)$ **11** $(-6, 60°)$

12 $(-8, -30°)$ **13** $(2, 28°)$ **14** $(5, 128°)$

15 $(-3, -2.6)$ **16** $(-4, 3.1)$

In exercises 17 to 24, find a pair of polar coordinates, to the nearest tenth, where $0 \le \theta < \pi$, for the points with the following rectangular coordinates.

17 $(2, 4)$ **18** $(3, 8)$ **19** $(-4, 2)$ **20** $(3, 0)$

21 $(-1, 0)$ **22** $(-2, -3)$ **23** $(5, -1)$ **24** $(3, -6)$

In exercises 25 to 30, write each equation in polar coordinate form.

25 $x = 8$ **26** $y = 3x$

27 $x - y = 16$ **28** $2y + x = 4$

29 $x^2 + y^2 = 25$ **30** $x^2 = y^2 + y + 1$

In exercises 31 to 42, write each equation in rectangular coordinate form.

31 $r^2 = 49$ **32** $r = 14$

33 $r \sin \theta = 5$ **34** $r \cos \theta = -7$

35 $5r \sin \theta + 6r \cos \theta = 1$ **36** $2r \sin \theta = 5 - r \sin \theta$

Set B

37 $r = 2 \cos \theta$ **38** $r = 3 \sec \theta$

39 $\tan \theta = 5$ **40** $r \sin \theta \tan \theta = 1$

41 $\theta = \frac{\pi}{6}$ **42** $\theta = -\frac{\pi}{2}$

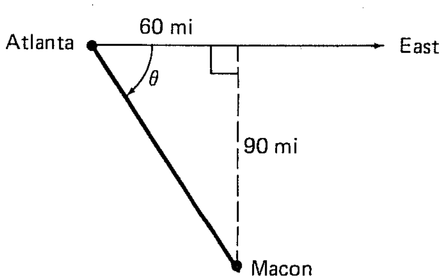


Figure 4-27

43 As shown in Fig. 4-27, Macon, GA, is located 60 mi east and 90 mi south of Atlanta. A weather station in Atlanta detects on their radar screen a severe storm centered over Macon. To the nearest mile, how far is the storm from the weather station? Find θ to the nearest degree.

44 In an experiment on orientation and navigation some homing pigeons were released 85 km from their loft. How many kilometers is the point of release west and how many kilometers north of the loft? (See Fig. 4-28.)

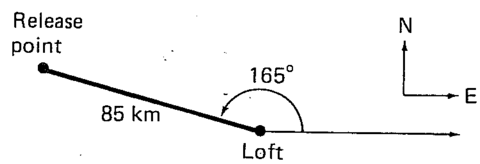


Figure 4-28

45 An explorer bee discovers a source of honey at noon (the time at which the bee uses ordinary polar coordinates for directions). The source is

located 800 m east and 1250 m south of the hive (Fig. 4-29). What polar coordinates will the bee signal for the other bees in the hive?

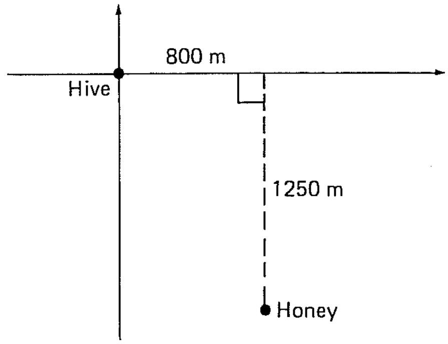


Figure 4-29

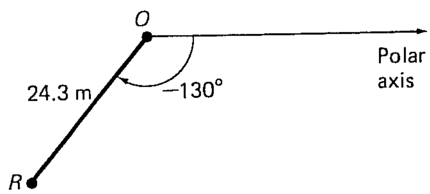


Figure 4-30

46 In a behavioral experiment, a turtle of a certain species was released at point O and observed resting at point R (see Fig. 4-30). Find the rectangular coordinates of R to the nearest tenth.

In exercises 47 to 50, plot at least 10 points (r, θ) that satisfy each equation. Then sketch each graph by joining these points.

■ **47** $r = 1$ ■ **48** $r = 3$ ■ **49** $\theta = 0$ ■ **50** $\theta = \frac{\pi}{4}$

Set C

51 The rectangular coordinates of P are (x, y), and the polar coordinates of P are (r, θ). Find polar coordinates of $(-x, y)$, $(x, -y)$, and $(-x, -y)$, where $r > 0$ and $0 \le \theta < 2\pi$.

52 The rectangular coordinates of P are (x, y), and the polar coordinates of P are (r, θ). Find the rectangular coordinates of $(-r, \theta)$, $(r, -\theta)$, and $(-r, -\theta)$.

53 Show that the distance d between the points (r_1, θ_1) and (r_2, θ_2) is given by the formula

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

54 A formula in some books is $\theta = \text{Arctan } y/x$, where (x, y) and (r, θ) are the respective rectangular and polar coordinates of point P. Under what conditions does this formula hold?

55 The slope of a line is 3 and its y intercept is -2. Find the equation of the line in polar form.

56 A circle centered at $(-3, 4)$ has radius 5. Find the equation of the circle in polar form.

4-7

GRAPHS OF POLAR EQUATIONS

The graph of an equation in x and y is the set of points (x, y) whose coordinates satisfy the equation. Similarly, the graph of an equation in polar coordinates r and θ is the set of points with polar coordinates that satisfy the equation.

EXAMPLE 1

(a) Graph $r = 2$. (b) Graph $\theta = \frac{\pi}{4}$

Solution

(a) The graph of $r = 2$ is the set of points with coordinates $(2, \theta)$, where θ may be any angle measure. Thus, the graph is a circle centered at O with radius 2. (See Fig. 4-31.)

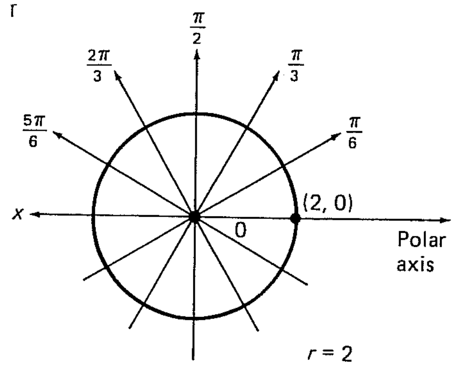


Figure 4-31

(b) The graph of $\theta = \frac{\pi}{4}$ is the set of points with coordinates $\left(r, \frac{\pi}{4}\right)$, where r may be any real number. Thus, the graph is a line through O that makes an angle of $\frac{\pi}{4}$ with the polar axis. (See Fig. 4-32.)

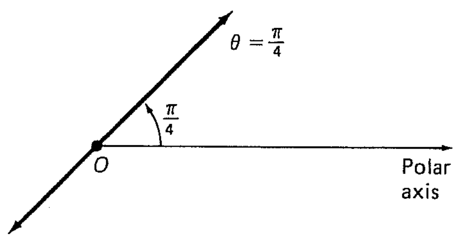


Figure 4-32

To graph a more complex equation, it is necessary to compute and graph a number of ordered pairs that satisfy the equation.

EXAMPLE 2

Graph $r = 2 \sin \theta$.

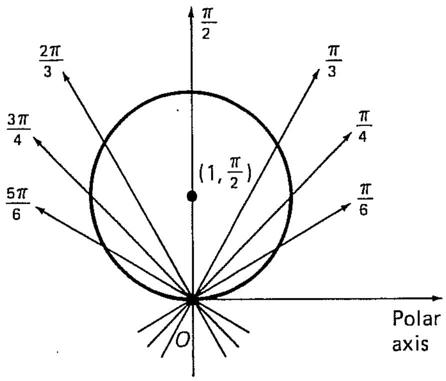


Figure 4-33

Solution

Find several ordered pairs that satisfy the above equation:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$

Since $\sin(\theta + \pi) = -\sin \theta$, a point represented by $(r, \theta + \pi) = (-2 \sin \theta, \theta + \pi)$ is the same point as $(r, \theta) = (2 \sin \theta, \theta)$. The values in the table are plotted in Fig. 4-33. The graph appears to be a circle centered at $\left(1, \frac{\pi}{2}\right)$ with radius 1.

EXAMPLE 3

Convert $r = 2 \sin \theta$ to rectangular coordinates and graph.

Solution

Substitute $\sqrt{x^2 + y^2}$ for r and $y/\sqrt{x^2 + y^2}$ for $\sin \theta$.

$$\sqrt{x^2 + y^2} = \frac{2y}{\sqrt{x^2 + y^2}}$$

Simplify

$$x^2 + y^2 = 2y \quad \text{or} \quad x^2 + y^2 - 2y = 0$$

Complete the square

$$x^2 + y^2 - 2y + 1 = 1$$
$$x^2 + (y - 1)^2 = 1$$

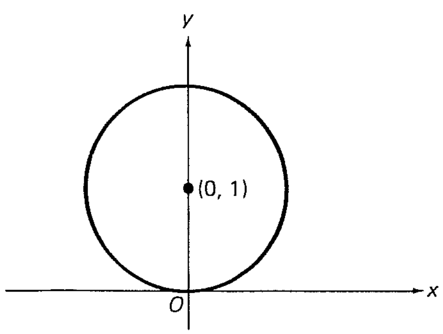


Figure 4-34

This is the equation of a circle centered at (0, 1) with radius 1. (See Fig. 4-34.)

Examples 2 and 3 illustrate that the graph of an equation in polar coordinates is identical to the graph of that equation converted to rectangular form.

EXAMPLE 4

Use a graphing utility to graph $r = 2 + \sin\theta$.

Solution

If you are using a computer graphics package, follow the directions for its use. We will illustrate the solution using a graphing calculator. We modify the program for the Casio fx-8000G in Appendix B, page 451, by editing it as follows.

Line 2	Range $-3, 3, 1, -3, 3, 1$
Line 4	$2 + \sin T \rightarrow R$
Line 9	$T \leq 2\pi \Rightarrow$ Goto 1

Line 9 makes use of the fact that $\sin(\theta + 2k\pi) = \sin\theta$, so values of θ (T in the program) between 0 and 2π are sufficient. Note, however, that it is not sufficient for the variable T to range from 0 to π, since the values of r for θ between π and 2π are not a repeat of those for θ between 0 and π. After editing the program, execute it. The resulting graph, called a cardioid, is illustrated in Figure 4-35.

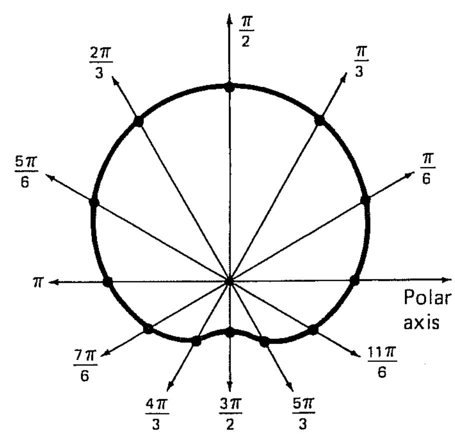


Figure 4-35

For some equations it is necessary to consider values of θ outside the interval 0 to 2π.

EXAMPLE 5 Graph $r = 2\cos 3\theta$.

As in Example 4, set up a table of values, plot the corresponding points, and join them to obtain the graph shown in Fig. 4-36. In this case, degrees are used for θ. The graph is the same if θ is in radians.

θ	3θ	$r = 2\cos 3\theta$
0°	0°	2
15°	45°	1.41
30°	90°	0
45°	135°	−1.41
60°	180°	−2
75°	225°	−1.41
90°	270°	0
105°	315°	1.41
120°	360°	2
135°	405°	1.41
150°	450°	0
165°	495°	−1.41
180°	540°	−2

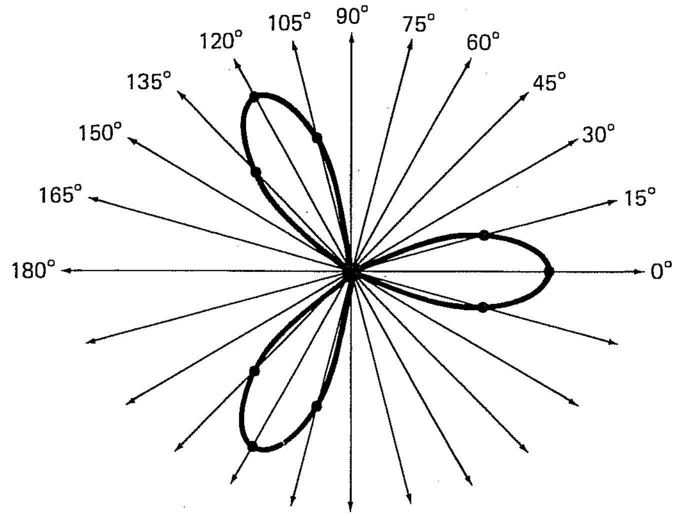


Figure 4-36

This graph is called a **three-leaved rose.**

Exercises

Set A

Graph each equation in exercises 1 to 20. Use a graphing utility to verify your graphs.

1 $r = 1$ **2** $r = -2$

3 $r = -3$ **4** $r = 3$

5 $\theta = -\frac{\pi}{4}$ **6** $\theta = 40°$

7 $\theta = 90°$ **8** $\theta = -\pi$

9 $r = \sin\theta$ **10** $r = \cos\theta$

11 $r = -2\cos\theta$ **12** $r = -3\sin\theta$

13 $r = \sin 3\theta$ **14** $r = \cos 2\theta$

15 $r = -2\cos 2\theta$ **16** $r = -3\sin 3\theta$

17 $r = 1 + \sin\theta$

18 $r = 1 + \cos\theta$

19 $r = 1 - 2\cos\theta$

20 $r = 1 - 2\sin\theta$

Set B

Graph each equation in exercises 21–32 with a graphing utility or by plotting points.

21 $r = \sin 4\theta$

22 $r = \cos 4\theta$

23 $r = -2\cos 5\theta$

24 $r = -2\sin 5\theta$

25 $r = \theta$

26 $r = -\theta$

27 $r = -2\theta$

28 $r = 2\theta$

29 $r = \sin\left(\theta + \frac{\pi}{3}\right)$

30 $r = \cos\left(\theta - \frac{\pi}{2}\right)$

31 $r = \sec\theta$

32 $r = \csc\theta$

Find a polar equation for each graph described in exercises 33 to 40.

33 A line through O with slope 1

34 A line through $\left(1, \frac{\pi}{2}\right)$ with slope -1

35 A line parallel to the one in exercise 33 through polar point $(-1, 0)$

36 A line perpendicular to the one in exercise 34 through $\left(2, \frac{\pi}{3}\right)$

37 A circle centered at O with radius 5

38 A circle centered at $\left(1, \frac{\pi}{4}\right)$ with radius 4

39 A parabola whose rectangular equation is $y = x^2$

40 A parabola whose rectangular equation is $x^2 - 1 = 2y$

Set C

In exercises 41 to 43, convert each equation to rectangular form and then graph the resulting equation.

41 $r = 6/(3\sin\theta + 2\cos\theta)$

42 $r = 6/(3\cos\theta + 2\sin\theta)$

43 $r = 6/(3\cos\theta - 2\sin\theta)$

44 Find a polar equation of the line with slope m that crosses the polar axis at $(k, 0)$.

45 Find a polar point of intersection of the graphs of $r = a\sin\theta + b$ and $r = c\sin\theta + d$. For what values of a, b, c, and d do the graphs have no point of intersection? Explain.

USING BASIC

Computing Area Under a Curve

Areas of regions enclosed by curves can be approximated by using randomly generated points. This approach is called the **Monte Carlo method**.

EXAMPLE

Write a program that uses the Monte Carlo method to find the area under the sine curve for the interval $0 \le x \le \frac{\pi}{2}$.

Analysis

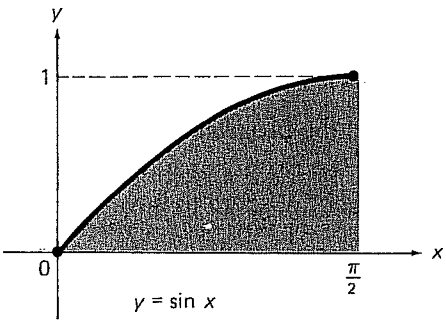


Figure 4-37

Graph $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$ and enclose it in a 1 by $\frac{\pi}{2}$ rectangle, as in Fig. 4-37. We need to find the area of the shaded region between the curve and the x axis. Our strategy is to write a program that generates ordered pairs (x, y) at random, where $0 < x < \frac{\pi}{2}$ and $0 < y < 1$ so that (x, y) is inside the rectangle. Each pair (x, y) is checked to determine if $y < \sin x$, that is, if (x, y) is under the sine curve. A record is kept of the number c of points under the sine curve and of the total number n of points randomly generated. Since the points were chosen at random,

$$\frac{c}{n} = \frac{\text{area under sine curve}}{\text{area of rectangle}}$$

Thus, over a large number of trials, the ratio $\frac{c}{n}$ times the area of the rectangle, $\frac{\pi}{2}$, will approximate the unknown area.

Program

DEF statement permits user definition of functions → 40

```
10  REM PROGRAM TO FIND AREA UNDER THE CURVE Y = F(X)
20  REM FOR X BETWEEN 0 AND (PI)/2
30  LET C = 0
40  DEF FNA(X) = SIN (X)
50  INPUT N
60  FOR K = 1 TO N
70  LET X = 1.5707963 * RND(1)
80  LET Y = RND(1)
90  IF Y > FNA(X) THEN 110
100  LET C = C + 1
110  NEXT K
120  LET A = (C/N) * 1.5707963
```

```
130  PRINT "THE AREA UNDER THE CURVE Y = F(X) FOR X"
140  PRINT "BETWEEN 0 AND (PI)/2 IS ABOUT ";
150  PRINT A
160  END
```

Output

```
]RUN
?10
THE AREA UNDER THE CURVE Y = F(X) FOR X
BETWEEN 0 AND (PI)/2 IS ABOUT .973893706

]RUN
?99
THE AREA UNDER THE CURVE Y = F(X) FOR X
BETWEEN 0 AND (PI)/2 IS ABOUT 1.16238926
```

Exercises

1 For each of the following values of N, run the program five times and record the results. For a given value of N, are the results exactly the same each time? What effect does increasing the size of N seem to have on your results?

(a) 10 (b) 25 (c) 100 (d) 200

2 Modify the program by replacing statement 40 by

```
40  DEF FNA(X) = 2*(COS(X))
```

(a) Graph $y = 2 \cos x$ for x between 0 and $\frac{\pi}{2}$ and shade the region bounded by the x axis and the graph for $0 \le x \le \frac{\pi}{2}$. Estimate the area of this region.

(b) Repeat exercise 1 using the revised program. Compare the results with your estimate in part (a).

3 Repeat exercise 2 by modifying statement 40 to use the following functions for $f(x)$:

(a) $y = \cos \frac{1}{2}x$ (b) $y = 3 \sin \frac{1}{2}x$ (c) $y = x$

(d) $y = x \sin x$ (e) $y = \mathrm{Tan}^{-1} x$ (f) $y = \mathrm{Tan}^{-1} 2x$

4 The equation of the unit circle is $x^2 + y^2 = 1$ and its area is $\pi(1^2) = \pi$. Write a program that uses the Monte Carlo method together with the unit circle to compute an approximation of π.

Chapter Summary and Review

Section 4-1

The inverse f^{-1} of a function f is the correspondence formed by reversing the pairings between the domain and range elements of f.

The graph of f^{-1} is the reflection image of the graph of f across the line $y = x$.

The inverse of the sine function is written $y = \sin^{-1} x$, which means $\sin y = x$. Similarly, $y = \cos^{-1} x$ means $\cos y = x$.

In exercises 1 to 3, write an equation for the inverse of each function where y is given in terms of x. Is the inverse a function?

1 $y = 2x + 1$ **2** $y = x^2 - 1$ **3** $y = 3/x$

In exercises 4 to 6, find all values of each expression in terms of radians.

4 $\sin^{-1}(-1)$ **5** $\cos^{-1} \sqrt{2}/2$ **6** $\sin^{-1} \sqrt{3}/2$

Section 4-2

The inverse sine function is written $y = \text{Sin}^{-1} x$, were $y = \text{Sin}^{-1} x$ means $y = \sin^{-1} x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

The inverse cosine function is written $y = \text{Cos}^{-1} x$, which means $y = \cos^{-1} x$ and $0 \le y \le \pi$.

In exercises 7 to 9, find the exact value in radians.

7 $\text{Sin}^{-1}(-1)$ **8** $\text{Cos}^{-1}(-\sqrt{2}/2)$ **9** $\sin(\text{Sin}^{-1} 0.74)$

Use your calculator set in radian mode to evaluate expressions 10 to 12 to four decimal places.

10 $\text{Cos}^{-1} 0.5732$ **11** $\text{Sin}^{-1}(-0.4369)$ **12** $\text{Cos}^{-1}(\cos 3.2450)$

Section 4-3

The inverse tangent function is written $y = \text{Tan}^{-1} x$, which means $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

The inverse cotangent function is written $y = \text{Cot}^{-1} x$, which means $\cot y = x$ and $0 < y < \pi$.

In exercises 13 to 15, find the exact value in radians.

13 $\text{Tan}^{-1}(-\sqrt{3})$ **14** $\text{Cot}^{-1} 1$ **15** $\tan(\text{Tan}^{-1} 7.63)$

Use your calculator set in radian mode to evaluate expressions 16 to 18 to four decimal places.

16 $\mathrm{Cot}^{-1}\, 6.4215$ **17** $\mathrm{Tan}^{-1}\,(-0.6557)$ **18** $\mathrm{Cot}^{-1}\,(\cot 5)$

Find the exact value of expressions 19 to 21 by referring to an angle in standard position.

19 $\cos\,(\mathrm{Tan}^{-1}\, 5/3)$ **20** $\sin\,(\mathrm{Cot}^{-1}\, 4)$

21 $\sec\,[\mathrm{Tan}^{-1}\,(-1/5)]$

Section 4-4

The inverse secant function is written $y = \mathrm{Sec}^{-1}\, x$, which means $\sec y = x$ and $0 \le y \le \pi$, $y \ne \dfrac{\pi}{2}$.

The inverse cosecant function is written $y = \mathrm{Csc}^{-1}\, x$, which means $\csc y = x$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$, $y \ne 0$.

Find the exact value of expressions 22 to 24 in radians.

22 $\mathrm{Sec}^{-1}\, 2\sqrt{3}/3$ **23** $\mathrm{Csc}^{-1}\,(-\sqrt{2})$ **24** $\sec\,[\mathrm{Csc}^{-1}\,(-2)]$

Section 4-5

A conditional equation is an equation that is true for some, but not all, allowable replacements of the variable. If such an equation contains trigonometric terms, it is called a conditional trigonometric equation.

Find exact solutions of equations 25 to 27 in the interval $0 \le x < 2\pi$.

25 $2\cos x - \sqrt{3} = 0$ **26** $2\sin^2 x + \sin x - 1 = 0$

27 $\sin 2x = \cos x$

Section 4-6

A polar coordinate system consists of a point O and a ray $\overrightarrow{OA}$ called the pole and polar axis, respectively.

Coordinates of a point P are specified according to the distance OP and the measure of the angle AOP (see Fig. 4-38).

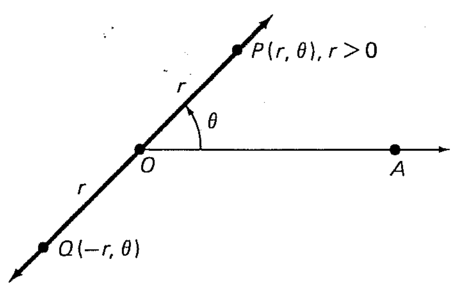


Figure 4-38

If point P has rectangular coordinates (x, y) and polar coordinates (r, θ), the following equations hold:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

In exercises 28 to 31, for the point with given polar coordinates, find a pair (r, θ) such that $0° \le \theta < 180°$.

28 $(3, -48°)$

29 $(-6, 19°)$

30 $(-5, -265°)$

31 $(1, 180°)$

In exercises 32 to 35, find the rectangular coordinates, to the nearest tenth, of the points with the following polar coordinates.

32 $\left(3, \dfrac{3\pi}{2}\right)$

33 $(-2, 55°)$

34 $(6.1, 2.1)$

35 $(-2, -1.5)$

Section 4-7
The graph of an equation in polar coordinates r and θ is the set of points with polar coordinates that satisfy the equation.

In exercises 36 to 38, graph each equation.

36 $r = 4$

37 $\theta = \dfrac{\pi}{3}$

38 $r = \sin 2\theta$

Chapter Test

1 Graph the inverse of the function graphed in Fig. 4-39.

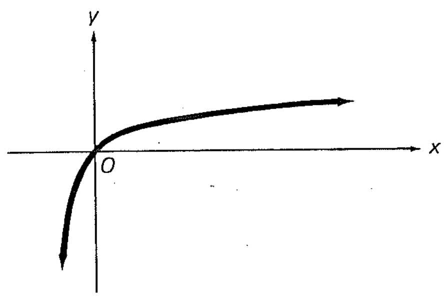


Figure 4-39

2 Write an equation where y is given in terms of x for the inverse of $y = -x^2 + 2$.

3 What are the domain and range of $y = -x^2 + 2$?

4 What are the domain and range of the inverse of $y = -x^2 + 2$?

5 Is the inverse of $y = -x^2 + 2$ a function?

6 Find all radian values of $\sin^{-1}(-\sqrt{2}/2)$.

7 Find the exact degree values of $\mathrm{Cos}^{-1}(-\sqrt{3}/2)$ and $\mathrm{Sin}^{-1}(-\sqrt{3}/2)$.

For items 8 to 10, use your calculator set in radian mode to evaluate to four decimal places.

8 $\mathrm{Tan}^{-1}\, 2.413$ **9** $\mathrm{Cot}^{-1}(-3.6411)$

10 $\mathrm{Sec}^{-1}\, 21$

Use an angle in standard position to evaluate items 11 to 13 exactly.

11 $\sin(\mathrm{Cos}^{-1}\, 1/4)$ **12** $\tan(\mathrm{Sec}^{-1}(-5/2))$

13 $\csc(\mathrm{Cot}^{-1}\, 3)$

In items 14 to 16, indicate the values of x for which the equations hold.

14 $\sin(\mathrm{Sin}^{-1}\, x) = x$ **15** $\mathrm{Tan}^{-1}(\tan x) = x$

16 $\mathrm{Sec}^{-1}(\sec x) = x$

In items 17 and 18, find all exact solutions in the interval $0 \leq x < 2\pi$.

17 $3\csc^2 x - 4 = 0$ **18** $\sin 4x = 0$

In items 19 and 20, use your calculator to find, to four decimal places, all solutions in the interval $0 \leq x < 2\pi$.

19 $4\sin(x - 3) + 3 = 0$ **20** $\sec(2x - 1) = 5$

21 The displacement y in inches of an air molecule by a simple sound is given in terms of time t (in seconds) by $y = 0.002 \sin 100\pi t$. Find the maximum displacement. At what values of t does the maximum displacement occur?

22 Find the first time t that the displacement in item 21 is 0.001 in.

In items 23 to 25, for the point with given polar coordinates, find a pair (r, θ) such that $0 \leq \theta < 180°$.

23 $(-3, -72°)$ **24** $(5, -15°)$ **25** $(-6, 296°)$

26 Write in polar form $y = 2x^2 - 3$.

27 Write in rectangular form $r = 2\sec\theta$.

28 Graph $r = 1 - \cos\theta$.

29 Find a polar equation for a line through $(1, \pi)$ with slope 2.

30 Find a polar equation for a circle centered at $(1, 0°)$ with radius 1.