127. Show that the period of $f(\theta) = \tan\theta$ is π.

128. Show that the period of $f(\theta) = \cot\theta$ is π.

129. Prove the reciprocal identities given in formula (2).

130. Prove the quotient identities given in formula (3).

131. Establish the identity:

$$(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta = 1$$

Discussion and Writing

132. Write down five properties of the tangent function. Explain the meaning of each.

133. Describe your understanding of the meaning of a periodic function.

134. Explain how to find the value of $\sin 390°$ using periodic properties.

135. Explain how to find the value of $\cos(-45°)$ using even–odd properties.

'Are You Prepared?' Answers

1. $\left\{x \mid x \neq -\frac{1}{2}\right\}$ **2.** Even **3.** False **4.** True

2.4 Graphs of the Sine and Cosine Functions*

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Graphing Techniques: Transformations (Section 1.6, pp. 57–66)

Now Work the 'Are You Prepared?' problems on page 146.

OBJECTIVES
1. Graph Functions of the Form $y = A\sin(\omega x)$ Using Transformations (p. 138)
2. Graph Functions of the Form $y = A\cos(\omega x)$ Using Transformations (p. 139)
3. Determine the Amplitude and Period of Sinusoidal Functions (p. 140)
4. Graph Sinusoidal Functions Using Key Points (p. 142)
5. Find an Equation for a Sinusoidal Graph (p. 145)

Since we want to graph the trigonometric functions in the xy-plane, we shall use the traditional symbols x for the independent variable (or argument) and y for the dependent variable (or value at x) for each function. So we write the six trigonometric functions as

$y = f(x) = \sin x$	$y = f(x) = \cos x$	$y = f(x) = \tan x$
$y = f(x) = \csc x$	$y = f(x) = \sec x$	$y = f(x) = \cot x$

Here the independent variable x represents an angle, measured in radians. In calculus, x will usually be treated as a real number. As we said earlier, these are equivalent ways of viewing x.

The Graph of the Sine Function $y = \sin x$

Since the sine function has period 2π, we need to graph $y = \sin x$ only on the interval $[0, 2\pi]$. The remainder of the graph will consist of repetitions of this portion of the graph.

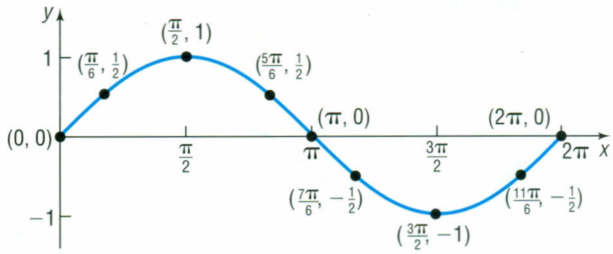
* For those who wish to include phase shifts here, Section 2.6 can be covered immediately after Section 2.4 without loss of continuity.

Table 6

x	$y = \sin x$	(x, y)
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
π	0	$(\pi, 0)$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
2π	0	$(2\pi, 0)$

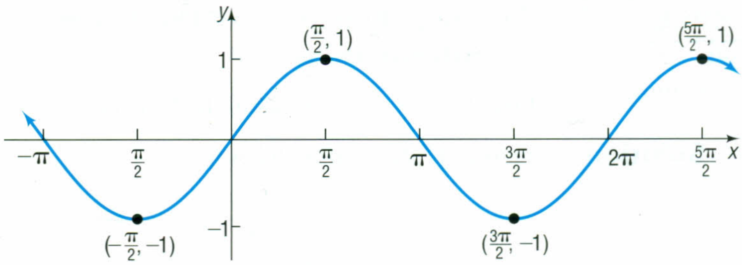
We begin by constructing Table 6, which lists some points on the graph of $y = \sin x$, $0 \le x \le 2\pi$. As the table shows, the graph of $y = \sin x$, $0 \le x \le 2\pi$, begins at the origin. As x increases from 0 to $\frac{\pi}{2}$, the value of $y = \sin x$ increases from 0 to 1; as x increases from $\frac{\pi}{2}$ to π to $\frac{3\pi}{2}$, the value of y decreases from 1 to 0 to -1; as x increases from $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0. If we plot the points listed in Table 6 and connect them with a smooth curve, we obtain the graph shown in Figure 44.

Figure 44
$y = \sin x, 0 \le x \le 2\pi$



The graph in Figure 44 is one period, or **cycle,** of the graph of $y = \sin x$. To obtain a more complete graph of $y = \sin x$, we continue the graph in each direction, as shown in Figure 45.

Figure 45
$y = \sin x, -\infty < x < \infty$



The graph of $y = \sin x$ illustrates some of the facts that we already know about the sine function.

Properties of the Sine Function $y = \sin x$

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The sine function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The sine function is periodic, with period 2π.
5. The x-intercepts are $\dots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$; the y-intercept is 0.
6. The maximum value is 1 and occurs at $x = \dots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$; the minimum value is -1 and occurs at $x = \dots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \dots$.

Now Work PROBLEMS **9, 11,** AND **13**

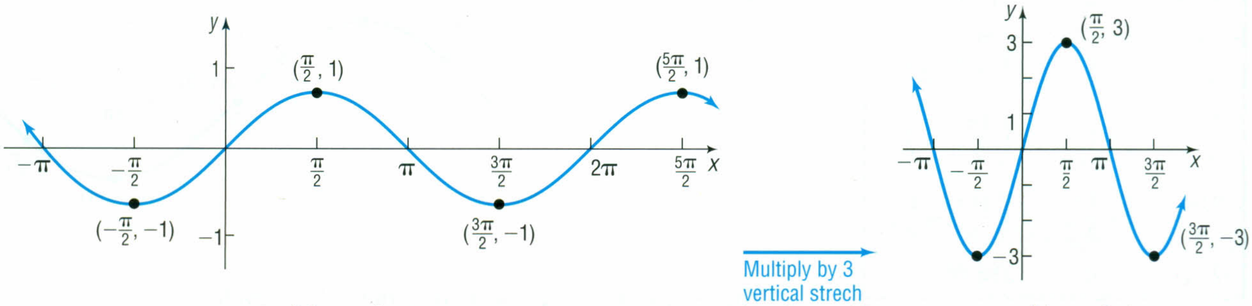
1 Graph Functions of the Form $y = A\sin(\omega x)$ Using Transformations

EXAMPLE 1 **Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations**

Graph $y = 3\sin x$ using transformations.

Solution Figure 46 illustrates the steps.

Figure 46



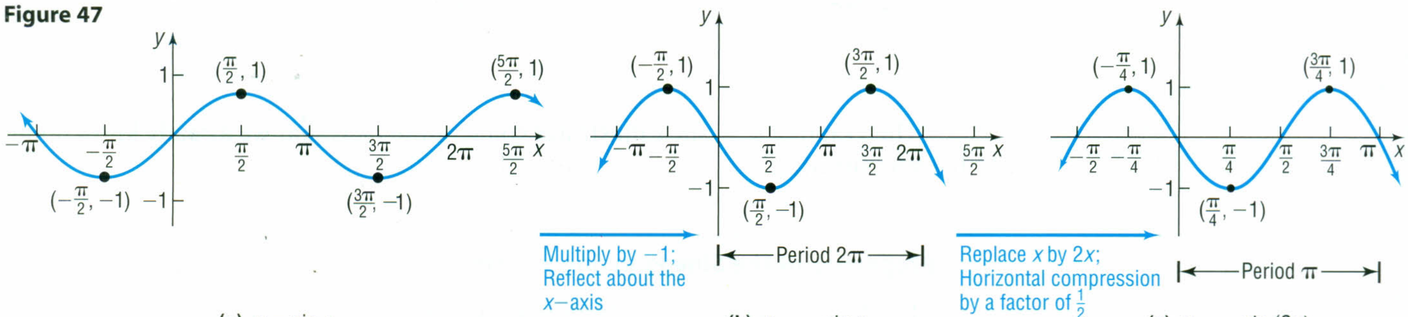
(a) $y = \sin x$ (b) $y = 3\sin x$

EXAMPLE 2 **Graphing Functions of the Form $y = A\sin(\omega x)$ Using Transformations**

Graph $y = -\sin(2x)$ using transformations.

Solution Figure 47 illustrates the steps.

Figure 47



(a) $y = \sin x$ (b) $y = -\sin x$ (c) $y = -\sin(2x)$

Notice in Figure 47(c) that the period of the function $y = -\sin(2x)$ is π due to the horizontal compression of the original period 2π by a factor of $\frac{1}{2}$.

Now Work PROBLEM 45 USING TRANSFORMATIONS

The Graph of the Cosine Function

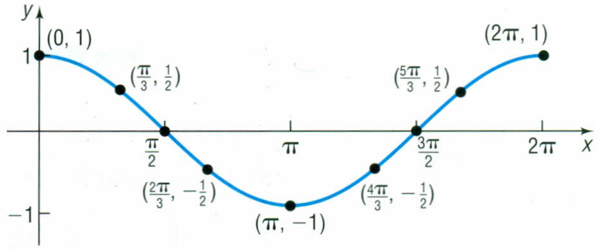
The cosine function also has period 2π. We proceed as we did with the sine function by constructing Table 7, which lists some points on the graph of $y = \cos x$, $0 \le x \le 2\pi$. As the table shows, the graph of $y = \cos x$, $0 \le x \le 2\pi$, begins at the point $(0, 1)$. As x increases from 0 to $\frac{\pi}{2}$ to π, the value of y decreases from 1 to 0 to -1; as x increases from π to $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0 to 1. As

Table 7

x	$y = \cos x$	(x, y)
0	1	(0, 1)
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3}, \frac{1}{2}\right)$
2π	1	$(2\pi, 1)$

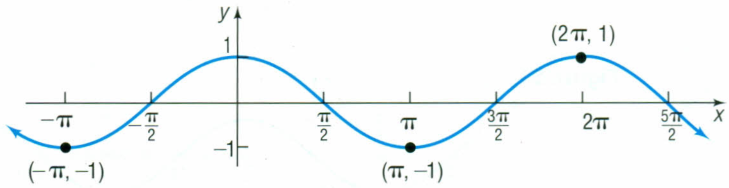
before, we plot the points in Table 7 to get one period or cycle of the graph. See Figure 48.

Figure 48
$y = \cos x, 0 \le x \le 2\pi$



A more complete graph of $y = \cos x$ is obtained by continuing the graph in each direction, as shown in Figure 49.

Figure 49
$y = \cos x, -\infty < x < \infty$



The graph of $y = \cos x$ illustrates some of the facts that we already know about the cosine function.

Properties of the Cosine Function

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The cosine function is an even function, as the symmetry of the graph with respect to the y-axis indicates.
4. The cosine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$; the y-intercept is 1.
6. The maximum value is 1 and occurs at $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$; the minimum value is -1 and occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$.

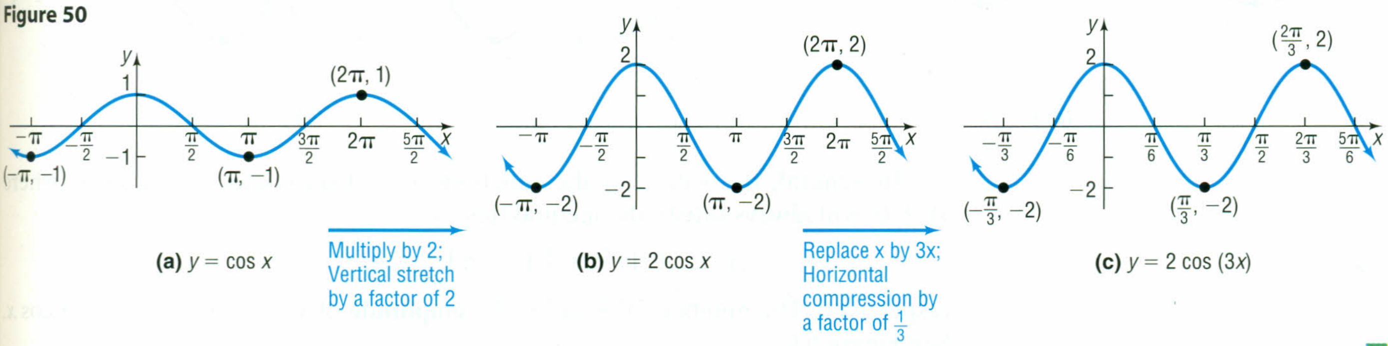
2 Graph Functions of the Form $y = A\cos(\omega x)$ Using Transformations

EXAMPLE 3 **Graphing Functions of the Form $y = A\cos(\omega x)$ Using Transformations**

Graph $y = 2\cos(3x)$ using transformations.

Solution Figure 50 shows the steps.

Figure 50



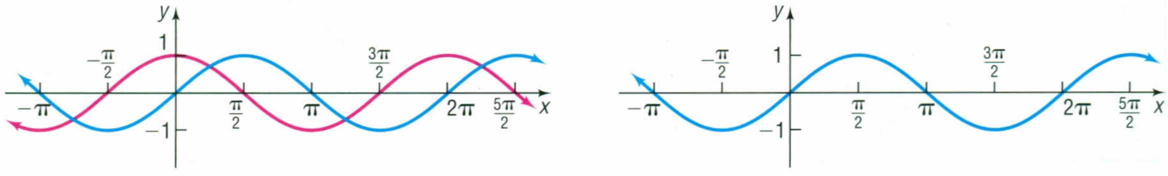
Notice in Figure 50(c) that the period of the function $y = 2\cos(3x)$ is $\frac{2\pi}{3}$ due to the compression of the original period 2π by a factor of $\frac{1}{3}$.

Now Work PROBLEM 53 USING TRANSFORMATIONS

Sinusoidal Graphs

Shift the graph of $y = \cos x$ to the right $\frac{\pi}{2}$ units to obtain the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$. See Figure 51(a). Now look at the graph of $y = \sin x$ in Figure 51(b). We see that the graph of $y = \sin x$ is the same as the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$.

Figure 51



(a) $y = \cos x$ $\quad y = \cos\left(x - \frac{\pi}{2}\right)$ **(b)** $y = \sin x$

Based on Figure 51, we conjecture that

$$\sin x = \cos\left(x - \frac{\pi}{2}\right)$$

Seeing the Concept

Graph $Y_1 = \sin x$ and $Y_2 = \cos\left(x - \frac{\pi}{2}\right)$. How many graphs do you see?

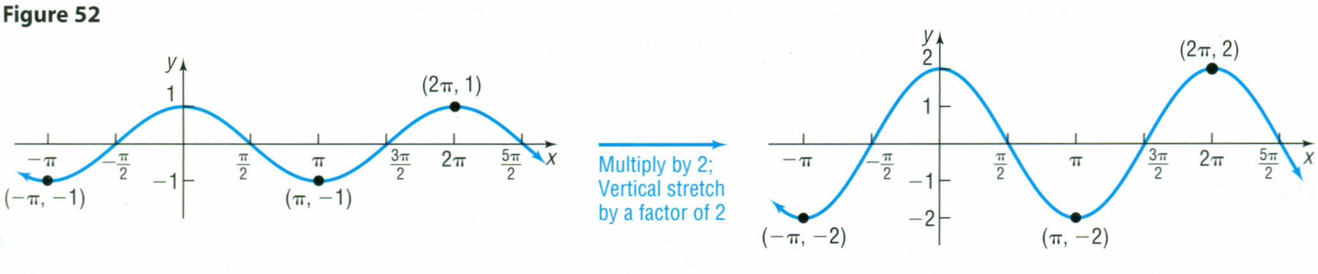
(We shall prove this fact in Chapter 3.) Because of this relationship, the graphs of functions of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ are referred to as **sinusoidal graphs.**

Let's look at some general properties of sinusoidal graphs.

3 Determine the Amplitude and Period of Sinusoidal Functions

In Figure 52(b) we show the graph of $y = 2\cos x$. Notice that the values of $y = 2\cos x$ lie between -2 and 2, inclusive.

Figure 52



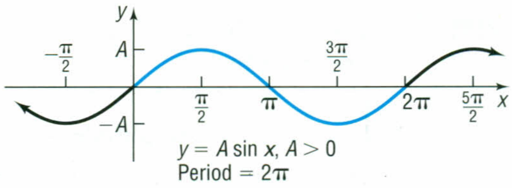
(a) $y = \cos x$ **(b)** $y = 2\cos x$

In general, the values of the functions $y = A\sin x$ and $y = A\cos x$, where $A \neq 0$, will always satisfy the inequalities

$$-|A| \leq A\sin x \leq |A| \quad \text{and} \quad -|A| \leq A\cos x \leq |A|$$

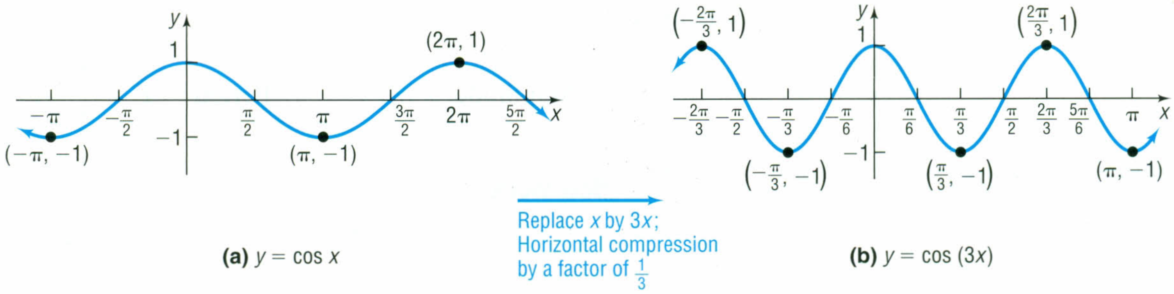
respectively. The number $|A|$ is called the **amplitude** of $y = A\sin x$ or $y = A\cos x$. See Figure 53.

Figure 53



In Figure 54(b), we show the graph of $y = \cos(3x)$. Notice that the period of this function is $\frac{2\pi}{3}$, due to the horizontal compression of the original period 2π by a factor of $\frac{1}{3}$.

Figure 54



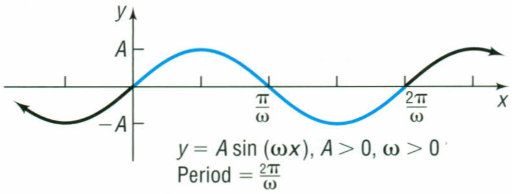
(a) $y = \cos x$ **(b)** $y = \cos(3x)$

In general, if $\omega > 0$, the functions $y = \sin(\omega x)$ and $y = \cos(\omega x)$ will have period $T = \frac{2\pi}{\omega}$. To see why, recall that the graph of $y = \sin(\omega x)$ is obtained from the graph of $y = \sin x$ by performing a horizontal compression or stretch by a factor $\frac{1}{\omega}$. This horizontal compression replaces the interval $[0, 2\pi]$, which contains one period of the graph of $y = \sin x$, by the interval $\left[0, \frac{2\pi}{\omega}\right]$, which contains one period of the graph of $y = \sin(\omega x)$. The period of the functions $y = \sin(\omega x)$ and $y = \cos(\omega x)$, $\omega > 0$, is $\frac{2\pi}{\omega}$.

For example, for the function $y = \cos(3x)$, graphed in Figure 54(b), $\omega = 3$, so the period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.

One period of the graph of $y = \sin(\omega x)$ or $y = \cos(\omega x)$ is called a **cycle.** Figure 55 illustrates the general situation. The blue portion of the graph is one cycle.

Figure 55



If $\omega < 0$ in $y = \sin(\omega x)$ or $y = \cos(\omega x)$, we use the Even–Odd Properties of the sine and cosine functions as follows:

$$\sin(-\omega x) = -\sin(\omega x) \quad \text{and} \quad \cos(-\omega x) = \cos(\omega x)$$

This gives us an equivalent form in which the coefficient of x in the argument is positive. For example,

$$\sin(-2x) = -\sin(2x) \quad \text{and} \quad \cos(-\pi x) = \cos(\pi x)$$

Because of this, we can assume $\omega > 0$.

THEOREM

If $\omega > 0$, the amplitude and period of $y = A\sin(\omega x)$ and $y = A\cos(\omega x)$ are given by

$$\text{Amplitude} = |A| \qquad \text{Period} = T = \frac{2\pi}{\omega} \tag{1}$$

EXAMPLE 4 **Finding the Amplitude and Period of a Sinusoidal Function**

Determine the amplitude and period of $y = 3\sin(4x)$.

Solution Comparing $y = 3\sin(4x)$ to $y = A\sin(\omega x)$, we find that $A = 3$ and $\omega = 4$. From equation (1),

$$\text{Amplitude} = |A| = 3 \qquad \text{Period} = T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Now Work PROBLEM 23

4 Graph Sinusoidal Functions Using Key Points

So far, we have graphed functions of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ using transformations. We now introduce another method that can be used to graph these functions.

Figure 56 shows one cycle of the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$. Notice that each graph consists of four parts corresponding to the four subintervals:

$$\left[0, \frac{\pi}{2}\right], \quad \left[\frac{\pi}{2}, \pi\right], \quad \left[\pi, \frac{3\pi}{2}\right], \quad \left[\frac{3\pi}{2}, 2\pi\right]$$

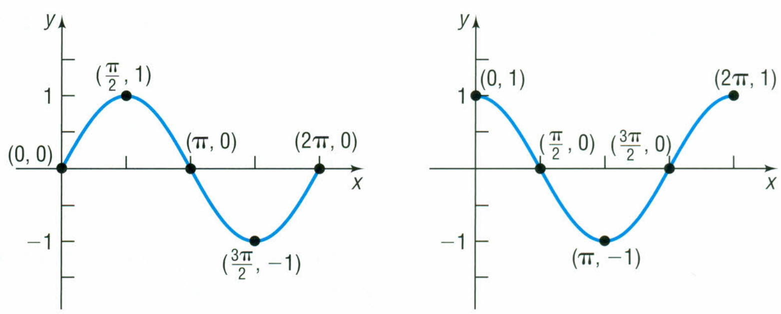
Each subinterval is of length $\frac{\pi}{2}$ (the period 2π divided by 4, the number of parts), and the endpoints of these intervals $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$, $x = 2\pi$ give rise to five key points on each graph:

For $y = \sin x$: $\quad (0, 0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right), (2\pi, 0)$

For $y = \cos x$: $\quad (0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$

Look again at Figure 56.

Figure 56



(a) $y = \sin x$ **(b)** $y = \cos x$

Steps for Graphing a Sinusoidal Function of the Form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ Using Key Points

STEP 1: Use the amplitude A to determine the maximum and minimum values of the function. This sets the scale for the y-axis.

STEP 2: Use the period $\frac{2\pi}{\omega}$ and divide the interval $\left[0, \frac{2\pi}{\omega}\right]$ into four subintervals of the same length.

STEP 3: Use the endpoints of these subintervals to obtain five key points on the graph.

STEP 4: Connect these points with a sinusoidal graph to obtain the graph of one cycle and extend the graph in each direction to make it complete.

EXAMPLE 5 Graphing a Sinusoidal Function Using Key Points

Graph: $y = 3\sin(4x)$

Solution Refer to Example 4. For $y = 3\sin(4x)$, the amplitude is 3 and the period is $\frac{\pi}{2}$. Because the amplitude is 3, the graph of $y = 3\sin(4x)$ will lie between -3 and 3 on the y-axis. Because the period is $\frac{\pi}{2}$, one cycle will begin at $x = 0$ and end at $x = \frac{\pi}{2}$.

We divide the interval $\left[0, \frac{\pi}{2}\right]$ into four subintervals, each of length $\frac{\pi}{2} \div 4 = \frac{\pi}{8}$, by finding the following values:

0	$0 + \frac{\pi}{8} = \frac{\pi}{8}$	$\frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$	$\frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$	$\frac{3\pi}{8} + \frac{\pi}{8} = \frac{\pi}{2}$
initial value	2nd value	3rd value	4th value	final value

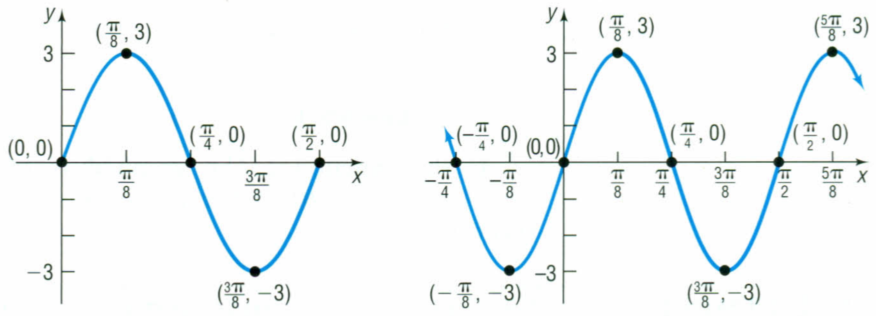
These values of x determine the x-coordinates of the five key points on the graph. To obtain the y-coordinates of the five key points of $y = 3\sin(4x)$, we multiply the y-coordinates of the five key points for $y = \sin x$ in Figure 56(a) by $A = 3$. The five key points are

$$(0, 0),\quad \left(\frac{\pi}{8}, 3\right),\quad \left(\frac{\pi}{4}, 0\right),\quad \left(\frac{3\pi}{8}, -3\right),\quad \left(\frac{\pi}{2}, 0\right)$$

NOTE We could also obtain the five key points by evaluating $y = 3\sin(4x)$ at each value of x.

We plot these five points and fill in the graph of the sine curve as shown in Figure 57(a). We extend the graph in either direction to obtain the complete graph shown in Figure 57(b).

Figure 57



(a) (b) $y = 3\sin(4x)$

COMMENT To graph a sinusoidal function of the form $y = A\sin(\omega x)$ or $y = A\cos(\omega x)$ using a graphing utility, we use the amplitude to set Ymin and Ymax and use the period to set Xmin and Xmax.

Check: Graph $y = 3\sin(4x)$ using transformations. Which graphing method do you prefer?

Now Work PROBLEM 45 USING KEY POINTS

EXAMPLE 6 Finding the Amplitude and Period of a Sinusoidal Function and Graphing It Using Key Points

Determine the amplitude and period of $y = 2\sin\left(-\frac{\pi}{2}x\right)$, and graph the function.

Solution Since the sine function is odd, we can use the equivalent form:

$$y = -2\sin\left(\frac{\pi}{2}x\right)$$

Comparing $y = -2\sin\left(\frac{\pi}{2}x\right)$ to $y = A\sin(\omega x)$, we find that $A = -2$ and $\omega = \frac{\pi}{2}$. The amplitude is $|A| = 2$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.

The graph of $y = -2\sin\left(\frac{\pi}{2}x\right)$ will lie between -2 and 2 on the y-axis. One cycle will begin at $x = 0$ and end at $x = 4$. We divide the interval $[0, 4]$ into four subintervals, each of length $4 \div 4 = 1$, by finding the following values:

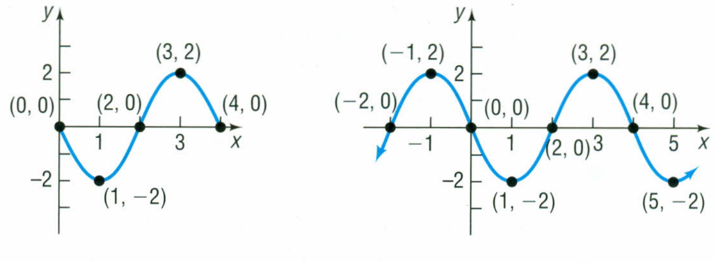
0	$0 + 1 = 1$	$1 + 1 = 2$	$2 + 1 = 3$	$3 + 1 = 4$
initial value	1st value	2nd value	3rd value	final value

Since $y = -2\sin\left(\frac{\pi}{2}x\right)$, we multiply the y-coordinates of the five key points in Figure 56(a) by $A = -2$. The five key points on the graph are

$$(0, 0), \quad (1, -2), \quad (2, 0), \quad (3, 2), \quad (4, 0)$$

We plot these five points and fill in the graph of the sine function as shown in Figure 58(a). Extending the graph in each direction, we obtain Figure 58(b).

Figure 58



(a) **(b)** $y = 2\sin\left(-\frac{\pi}{2}x\right)$

Check: Graph $y = 2\sin\left(-\frac{\pi}{2}x\right)$ using transformations. Which graphing method do you prefer?

Now Work PROBLEM 49 USING KEY POINTS

If the function to be graphed is of the form $y = A\sin(\omega x) + B$ [or $y = A\cos(\omega x) + B$], first graph $y = A\sin(\omega x)$ [or $y = A\cos(\omega x)$] and then use a vertical shift.

EXAMPLE 7 Finding the Amplitude and Period of a Sinusoidal Function and Graphing It Using Key Points

Determine the amplitude and period of $y = -4\cos(\pi x) - 2$, and graph the function.

Solution We begin by graphing the function $y = -4\cos(\pi x)$. Comparing $y = -4\cos(\pi x)$ with $y = A\cos(\omega x)$, we find that $A = -4$ and $\omega = \pi$. The amplitude is $|A| = |-4| = 4$, and the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.

The graph of $y = -4\cos(\pi x)$ will lie between -4 and 4 on the y-axis. One cycle will begin at $x = 0$ and end at $x = 2$. We divide the interval $[0, 2]$ into four subintervals, each of length $2 \div 4 = \frac{1}{2}$, by finding the following values:

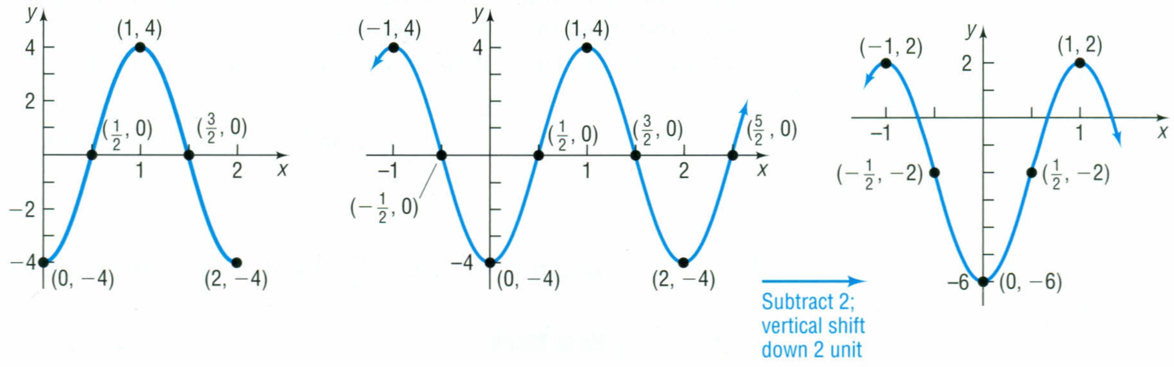
$$\underset{\text{initial value}}{0} \qquad \underset{\text{1st value}}{0 + \frac{1}{2} = \frac{1}{2}} \qquad \underset{\text{2nd value}}{\frac{1}{2} + \frac{1}{2} = 1} \qquad \underset{\text{3rd value}}{1 + \frac{1}{2} = \frac{3}{2}} \qquad \underset{\text{final value}}{\frac{3}{2} + \frac{1}{2} = 2}$$

Since $y = -4\cos(\pi x)$, we multiply the y-coordinates of the five key points of $y = \cos x$ shown in Figure 56(b) by $A = -4$ to obtain the five key points on the graph of $y = -4\cos(\pi x)$:

$$(0, -4), \quad \left(\frac{1}{2}, 0\right), \quad (1, 4), \quad \left(\frac{3}{2}, 0\right), \quad (2, -4)$$

We plot these five points and fill in the graph of the cosine function as shown in Figure 59(a). Extending the graph in each direction, we obtain Figure 59(b), the graph of $y = -4\cos(\pi x)$. A vertical shift down 2 units gives the graph of $y = -4\cos(\pi x) - 2$, as shown in Figure 59(c).

Figure 59



(a) **(b)** $y = -4\cos(\pi x)$ **(c)** $y = -4\cos(\pi x) - 2$

Now Work PROBLEM 59

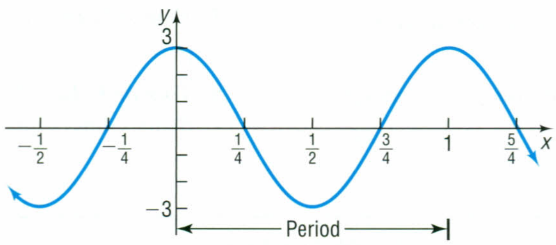
5 Find an Equation for a Sinusoidal Graph

We can also use the ideas of amplitude and period to identify a sinusoidal function when its graph is given.

EXAMPLE 8 Finding an Equation for a Sinusoidal Graph

Find an equation for the graph shown in Figure 60.

Figure 60



Solution The graph has the characteristics of a cosine function. Do you see why? So we view the equation as a cosine function $y = A\cos(\omega x)$ with $A = 3$ and period $T = 1$. Then $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. The cosine function whose graph is given in Figure 60 is

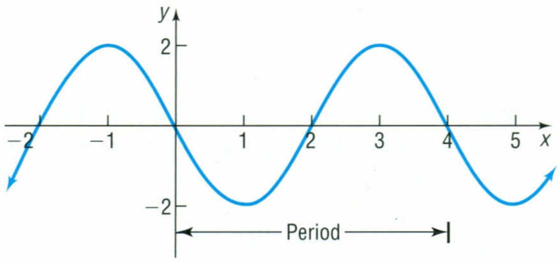
$$y = A\cos(\omega x) = 3\cos(2\pi x)$$

Check: Graph $Y_1 = 3\cos(2\pi x)$ and compare the result with Figure 60.

EXAMPLE 9 **Finding an Equation for a Sinusoidal Graph**

Find an equation for the graph shown in Figure 61.

Figure 61



Solution The graph is sinusoidal, with amplitude $|A| = 2$. The period is 4, so $\frac{2\pi}{\omega} = 4$ or $\omega = \frac{\pi}{2}$. Since the graph passes through the origin, it is easiest to view the equation as a sine function,* but notice that the graph is actually the reflection of a sine function about the x-axis (since the graph is decreasing near the origin). This requires that $A = -2$. The sine function whose graph is given in Figure 61 is

$$y = A\sin(\omega x) = -2\sin\left(\frac{\pi}{2}x\right)$$

Check: Graph $Y_1 = -2\sin\left(\frac{\pi}{2}x\right)$ and compare the result with Figure 61.

Now Work PROBLEMS 67 AND 71

2.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Use transformations to graph $y = 3x^2$ (pp. 57–66)
2. Use transformations to graph $y = -x^2$. (pp. 57–66)

Concepts and Vocabulary

3. The maximum value of $y = \sin x, 0 \le x \le 2\pi$, is ______ and occurs at $x =$ ______.
4. The function $y = A\sin(\omega x)$, $A > 0$, has amplitude 3 and period 2; then $A =$ ______ and $\omega =$ ______.
5. The function $y = 3\cos(6x)$ has amplitude ______ and period ______.
6. ***True or False*** The graphs of $y = \sin x$ and $y = \cos x$ are identical except for a horizontal shift.
7. ***True or False*** For $y = 2\sin(\pi x)$, the amplitude is 2 and the period is $\frac{\pi}{2}$.
8. ***True or False*** The graph of the sine function has infinitely many x-intercepts.

Skill Building

In Problems 9–18, if necessary, refer to a graph to answer each question.

9. What is the y-intercept of $y = \sin x$?
10. What is the y-intercept of $y = \cos x$?
11. For what numbers x, $-\pi \le x \le \pi$, is the graph of $y = \sin x$ increasing?
12. For what numbers x, $-\pi \le x \le \pi$, is the graph of $y = \cos x$ decreasing?

* The equation could also be viewed as a cosine function with a horizontal shift, but viewing it as a sine function is easier.

13. What is the largest value of $y = \sin x$?

14. What is the smallest value of $y = \cos x$?

15. For what numbers x, $0 \le x \le 2\pi$, does $\sin x = 0$?

16. For what numbers x, $0 \le x \le 2\pi$, does $\cos x = 0$?

17. For what numbers x, $-2\pi \le x \le 2\pi$, does $\sin x = 1$? Where does $\sin x = -1$?

18. For what numbers x, $-2\pi \le x \le 2\pi$, does $\cos x = 1$? Where does $\cos x = -1$?

In Problems 19–28, determine the amplitude and period of each function without graphing.

19. $y = 2\sin x$

20. $y = 3\cos x$

21. $y = -4\cos(2x)$

22. $y = -\sin\left(\frac{1}{2}x\right)$

23. $y = 6\sin(\pi x)$

24. $y = -3\cos(3x)$

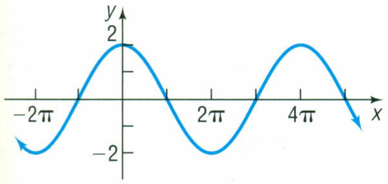
25. $y = -\frac{1}{2}\cos\left(\frac{3}{2}x\right)$

26. $y = \frac{4}{3}\sin\left(\frac{2}{3}x\right)$

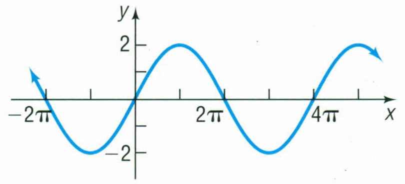
27. $y = \frac{5}{3}\sin\left(-\frac{2\pi}{3}x\right)$

28. $y = \frac{9}{5}\cos\left(-\frac{3\pi}{2}x\right)$

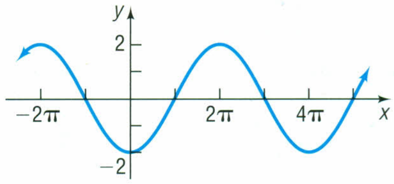
In Problems 29–38, match the given function to one of the graphs (A)–(J).



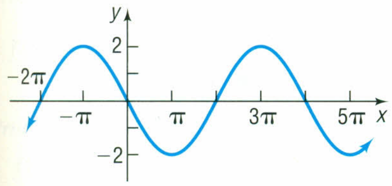
(A)



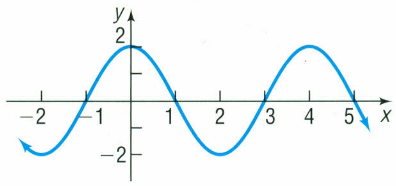
(B)



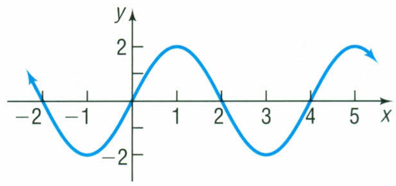
(C)



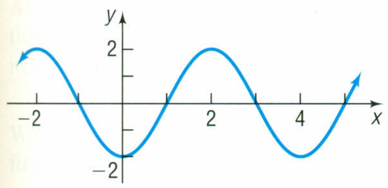
(D)



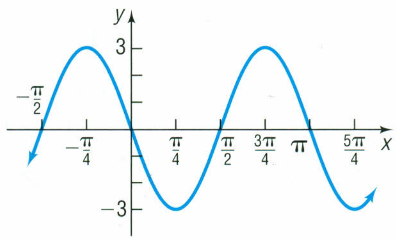
(E)



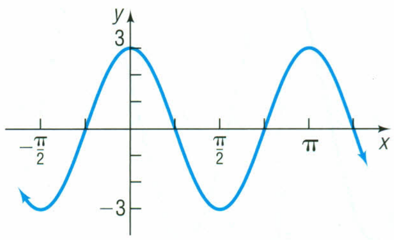
(F)



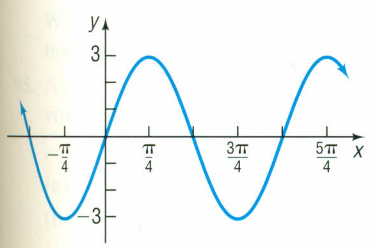
(G)



(H)



(I)



(J)

29. $y = 2\sin\left(\frac{\pi}{2}x\right)$

30. $y = 2\cos\left(\frac{\pi}{2}x\right)$

31. $y = 2\cos\left(\frac{1}{2}x\right)$

32. $y = 3\cos(2x)$

33. $y = -3\sin(2x)$

34. $y = 2\sin\left(\frac{1}{2}x\right)$

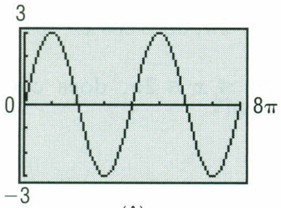
35. $y = -2\cos\left(\frac{1}{2}x\right)$

36. $y = -2\cos\left(\frac{\pi}{2}x\right)$

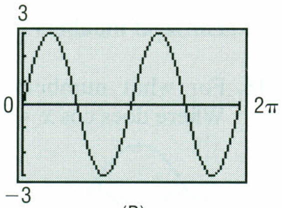
37. $y = 3\sin(2x)$

38. $y = -2\sin\left(\frac{1}{2}x\right)$

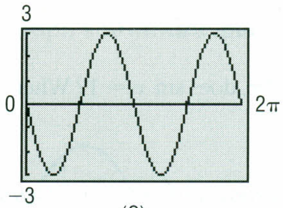
In Problems 39–42, match the given function to one of the graphs (A)–(D).



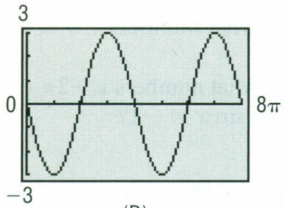
(A)



(B)



(C)



(D)

39. $y = 3\sin\left(\frac{1}{2}x\right)$ **40.** $y = -3\sin(2x)$ **41.** $y = 3\sin(2x)$ **42.** $y = -3\sin\left(\frac{1}{2}x\right)$

In Problems 43–66, graph each function. Be sure to label key points and show at least two cycles.

43. $y = 4\cos x$ **44.** $y = 3\sin x$ **45.** $y = -4\sin x$ **46.** $y = -3\cos x$

47. $y = \cos(4x)$ **48.** $y = \sin(3x)$ **49.** $y = \sin(-2x)$ **50.** $y = \cos(-2x)$

51. $y = 2\sin\left(\frac{1}{2}x\right)$ **52.** $y = 2\cos\left(\frac{1}{4}x\right)$ **53.** $y = -\frac{1}{2}\cos(2x)$ **54.** $y = -4\sin\left(\frac{1}{8}x\right)$

55. $y = 2\sin x + 3$ **56.** $y = 3\cos x + 2$ **57.** $y = 5\cos(\pi x) - 3$ **58.** $y = 4\sin\left(\frac{\pi}{2}x\right) - 2$

59. $y = -6\sin\left(\frac{\pi}{3}x\right) + 4$ **60.** $y = -3\cos\left(\frac{\pi}{4}x\right) + 2$ **61.** $y = 5 - 3\sin(2x)$ **62.** $y = 2 - 4\cos(3x)$

63. $y = \frac{5}{3}\sin\left(-\frac{2\pi}{3}x\right)$ **64.** $y = \frac{9}{5}\cos\left(-\frac{3\pi}{2}x\right)$ **65.** $y = -\frac{3}{2}\cos\left(\frac{\pi}{4}x\right) + \frac{1}{2}$ **66.** $y = -\frac{1}{2}\sin\left(\frac{\pi}{8}x\right) + \frac{3}{2}$

In Problems 67–70, write the equation of a sine function that has the given characteristics.

67. Amplitude: 3
Period: π

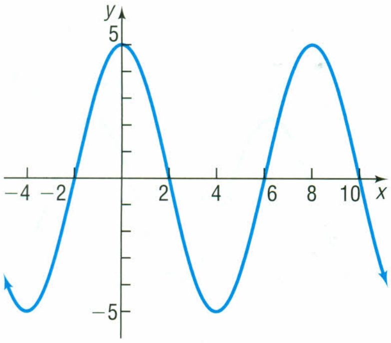
68. Amplitude: 2
Period: 4π

69. Amplitude: 3
Period: 2

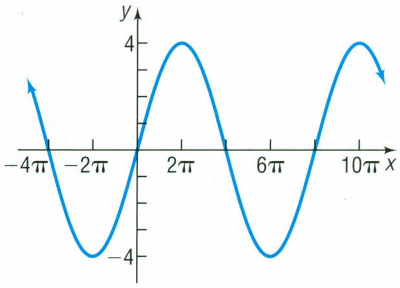
70. Amplitude: 4
Period: 1

In Problems 71–84, find an equation for each graph.

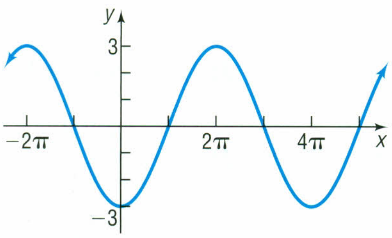
71.



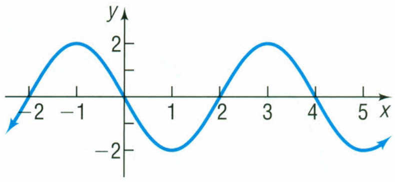
72.



73.



74.



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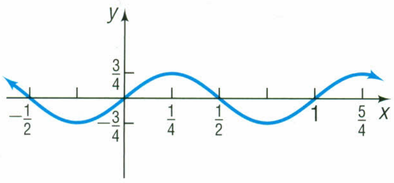

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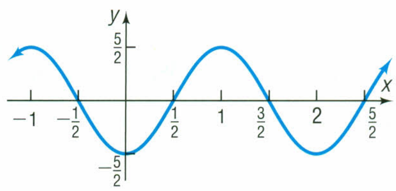

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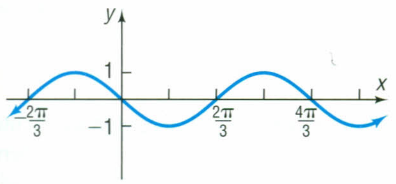
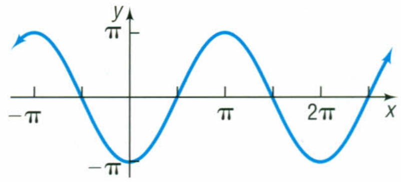


78.



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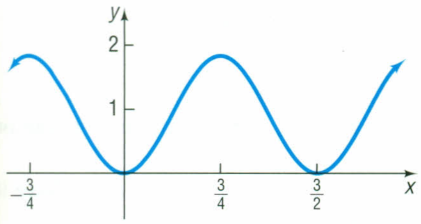
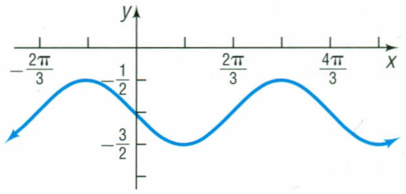
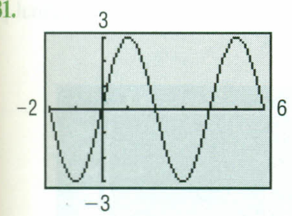


80.



81.



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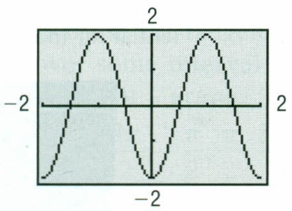
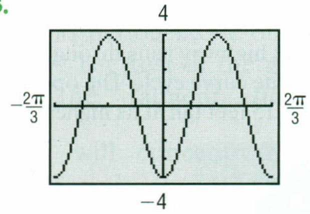
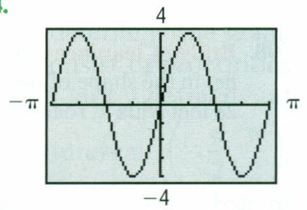


83.



84.



Applications and Extensions

In Problems 85–88, find the average rate of change of f from 0 to $\frac{\pi}{2}$.

85. $f(x) = \sin x$ **86.** $f(x) = \cos x$ **87.** $f(x) = \sin\left(\frac{x}{2}\right)$ **88.** $f(x) = \cos(2x)$

In Problems 89–92, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and graph each of these functions.

89. $f(x) = \sin x$, $g(x) = 4x$

90. $f(x) = \cos x$, $g(x) = \frac{1}{2}x$

91. $f(x) = -2x$, $g(x) = \cos x$

92. $f(x) = -3x$, $g(x) = \sin x$

93. Alternating Current (ac) Circuits The current I, in amperes, flowing through an ac (alternating current) circuit at time t in seconds, is

$$I(t) = 220\sin(60\pi t) \qquad t \geq 0$$

What is the period? What is the amplitude? Graph this function over two periods.

94. Alternating Current (ac) Circuits The current I, in amperes, flowing through an ac (alternating current) circuit at time t, in seconds, is

$$I(t) = 120\sin(30\pi t) \qquad t \geq 0$$

What is the period? What is the amplitude? Graph this function over two periods.

95. Alternating Current (ac) Generators The voltage V, in volts, produced by an ac generator at time t, in seconds, is

$$V(t) = 220\sin(120\pi t)$$

(a) What is the amplitude? What is the period?

(b) Graph V over two periods, beginning at $t = 0$.

(c) If a resistance of $R = 10$ ohms is present, what is the current I?

[**Hint:** Use Ohm's Law, $V = IR$.]

(d) What is the amplitude and period of the current I?

(e) Graph I over two periods, beginning at $t = 0$.

96. Alternating Current (ac) Generators The voltage V, in volts, produced by an ac generator at time t, in seconds, is

$$V(t) = 120\sin(120\pi t)$$

(a) What is the amplitude? What is the period?

(b) Graph V over two periods, beginning at $t = 0$.

(c) If a resistance of $R = 20$ ohms is present, what is the current I?

[**Hint:** Use Ohm's Law, $V = IR$.]

(d) What is the amplitude and period of the current I?

(e) Graph I over two periods, beginning at $t = 0$.

97. Alternating Current (ac) Generators The voltage V produced by an ac generator is sinusoidal. As a function of time, the voltage V is

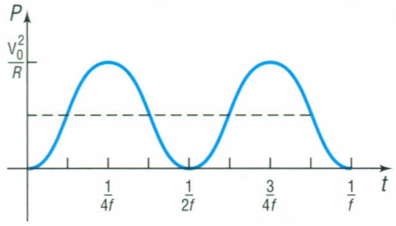
$$V(t) = V_0\sin(2\pi f t)$$

where f is the **frequency,** the number of complete oscillations (cycles) per second. [In the United States and Canada, f is 60 hertz (Hz).] The **power** P delivered to a resistance R at any time t is defined as

$$P(t) = \frac{[V(t)]^2}{R}$$

(a) Show that $P(t) = \frac{V_0^2}{R}\sin^2(2\pi f t)$.

(b) The graph of P is shown in the figure. Express P as a sinusoidal function.

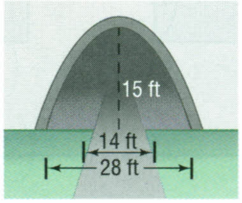


Power in an ac generator

(c) Deduce that

$$\sin^2(2\pi f t) = \frac{1}{2}[1 - \cos(4\pi f t)]$$

98. Bridge Clearance A one-lane highway runs through a tunnel in the shape of one-half a sine curve cycle. The opening is 28 feet wide at road level and is 15 feet tall at its highest point.



(a) Find an equation for the sine curve that fits the opening. Place the origin at the left end of the sine curve.

(b) If the road is 14 feet wide with 7-foot shoulders on each side, what is the height of the tunnel at the edge of the road?

Sources: *en.wikipedia.org/wiki/Interstate_Highway_standards* and *Ohio Revised Code*

99. Biorhythms In the theory of biorhythms, a sine function of the form

$$P(t) = 50\sin(\omega t) + 50$$

is used to measure the percent P of a person's potential at time t, where t is measured in days and $t = 0$ is the person's birthday. Three characteristics are commonly measured:

Physical potential: period of 23 days

Emotional potential: period of 28 days

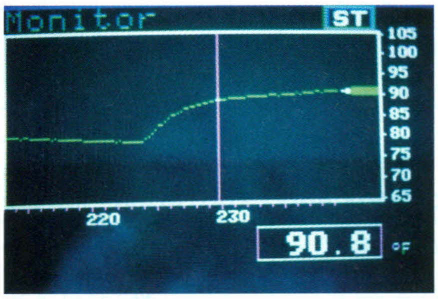
Intellectual potential: period of 33 days

(a) Find ω for each characteristic.

(b) Using a graphing utility, graph all three functions on the same screen.

(c) Is there a time t when all three characteristics have 100% potential? When is it?

(d) Suppose that you are 20 years old today ($t = 7305$ days). Describe your physical, emotional, and intellectual potential for the next 30 days.



100. Graph $y = |\cos x|, -2\pi \le x \le 2\pi$.

101. Graph $y = |\sin x|, -2\pi \le x \le 2\pi$.

Discussion and Writing

102. Explain how you would scale the x-axis and y-axis before graphing $y = 3\cos(\pi x)$.

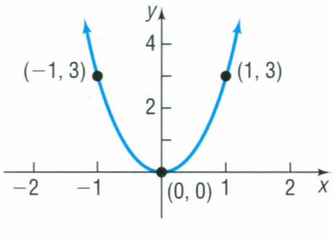
103. Explain the term *amplitude* as it relates to the graph of a sinusoidal function.

104. Explain how the amplitude and period of a sinusoidal graph are used to establish the scale on each coordinate axis.

105. Find an application in your major field that leads to a sinusoidal graph. Write a paper about your findings.

'Are You Prepared?' Answers

1. Vertical stretch by a factor of 3



2. Reflection about the x-axis