2.5 Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Vertical Asymptotes (Section 1.6, pp. 66–67)

Now Work the 'Are You Prepared?' problems on page 156.

OBJECTIVES
1 Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$ (p. 153)
2 Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$ (p. 155)

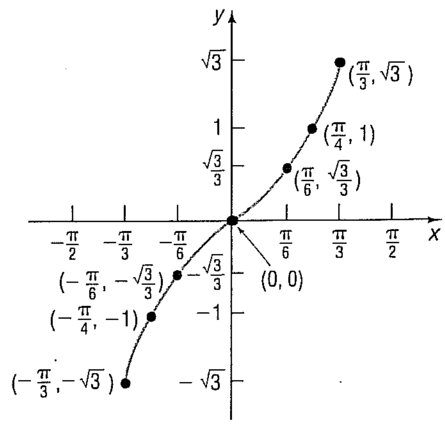
The Graph of the Tangent Function

Because the tangent function has period π, we only need to determine the graph over some interval of length π. The rest of the graph will consist of repetitions of that graph. Because the tangent function is not defined at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, we will concentrate on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, of length π, and construct Table 8, which lists some points on the graph of $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. We plot the points in the table and connect them with a smooth curve. See Figure 62 for a partial graph of $y = \tan x$, where $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

Table 8

x	$y = \tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.73$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -0.58$	$\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx 0.58$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$

Figure 62
$y = \tan x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$



To complete one period of the graph of $y = \tan x$, we need to investigate the behavior of the function as x approaches $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We must be careful, though, because $y = \tan x$ is not defined at these numbers. To determine this behavior, we use the identity

$$\tan x = \frac{\sin x}{\cos x}$$

See Table 9. If x is close to $\frac{\pi}{2} \approx 1.5708$, but remains less than $\frac{\pi}{2}$, then $\sin x$ will be close to 1 and $\cos x$ will be positive and close to 0. (To see this, refer back to the

graphs of the sine function and the cosine function.) So the ratio $\frac{\sin x}{\cos x}$ will be po-itive and large. In fact, the closer x gets to $\frac{\pi}{2}$, the closer $\sin x$ gets to 1 and $\cos x$ ge to 0, so $\tan x$ approaches $\infty \left(\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty \right)$. In other words, the vertical lin $x = \frac{\pi}{2}$ is a vertical asymptote to the graph of $y = \tan x$.

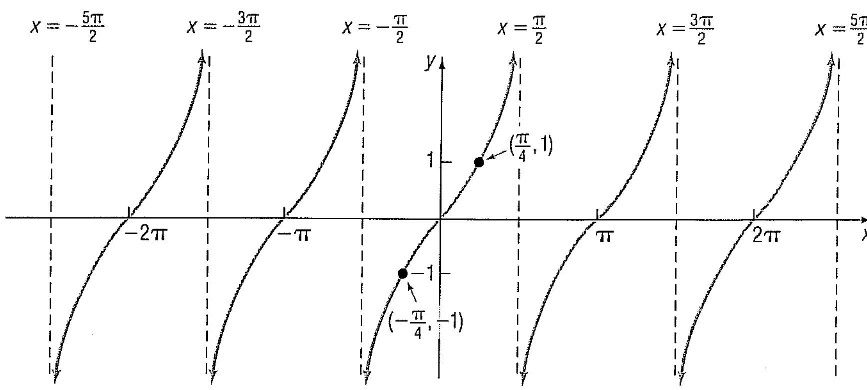
Table 9

x	$\sin x$	$\cos x$	$y = \tan x$
$\frac{\pi}{3} \approx 1.05$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3} \approx 1.73$
1.5	0.9975	0.0707	14.1
1.57	0.9999	$7.96E^{-4}$	1255.8
1.5707	0.9999	$9.6E^{-5}$	10,381
$\frac{\pi}{2} \approx 1.5708$	1	0	Undefined

If x is close to $-\frac{\pi}{2}$, but remains greater than $-\frac{\pi}{2}$, then $\sin x$ will be close to − and $\cos x$ will be positive and close to 0. The ratio $\frac{\sin x}{\cos x}$ approache $-\infty \left(\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty \right)$. In other words, the vertical line $x = -\frac{\pi}{2}$ is also a vertica asymptote to the graph.

With these observations, we can complete one period of the graph. We obtai the complete graph of $y = \tan x$ by repeating this period, as shown in Figure 63.

Figure 63
$y = \tan x$, $-\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$



Check: Graph $Y_1 = \tan x$ and compare the result with Figure 63. Use TRACE to see what happens as x gets close to $\frac{\pi}{2}$, but is less than $\frac{\pi}{2}$. Be sure to set the WINDOW accordingly and to use DOT mode.

The graph of $y = \tan x$ in Figure 63 illustrates the following properties.

Properties of the Tangent Function

1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.

(continued)

4. The tangent function is periodic, with period π.
5. The x-intercepts are $\dots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$.

Now Work PROBLEMS 7 AND 15

1 Graph Functions of the Form $y = A\tan(\omega x) + B$ and $y = A\cot(\omega x) + B$

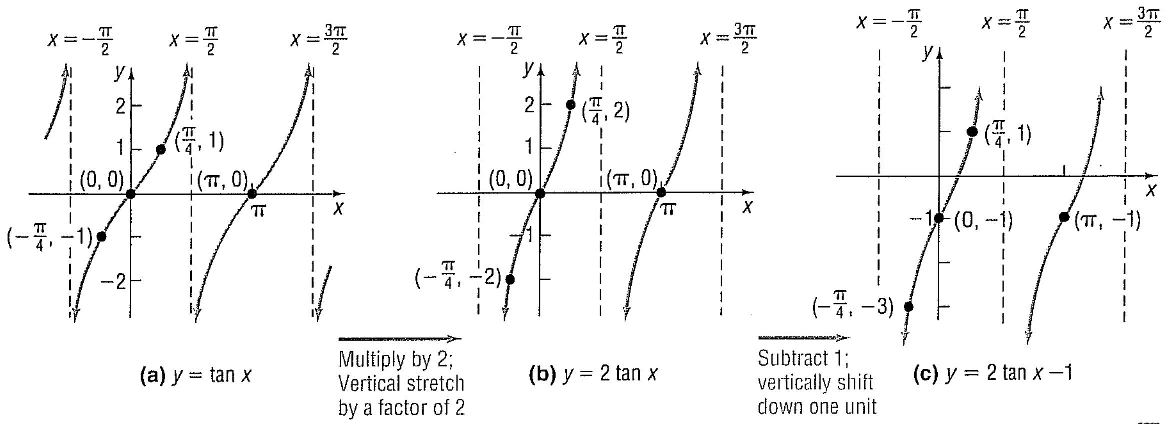
For tangent functions, there is no concept of amplitude since the range of the tangent function is $(-\infty, \infty)$. The role of A in $y = A\tan(\omega x) + B$ is to provide the magnitude of the vertical stretch. The period of $y = \tan x$ is π, so the period of $y = A\tan(\omega x) + B$ is $\frac{\pi}{\omega}$, caused by the horizontal compression of the graph by a factor of $\frac{1}{\omega}$. Finally, the presence of B indicates that a vertical shift is required.

EXAMPLE 1 Graphing Functions of the Form $y = A\tan(\omega x) + B$

Graph: $y = 2\tan x - 1$

Solution Figure 64 shows the steps using transformations.

Figure 64

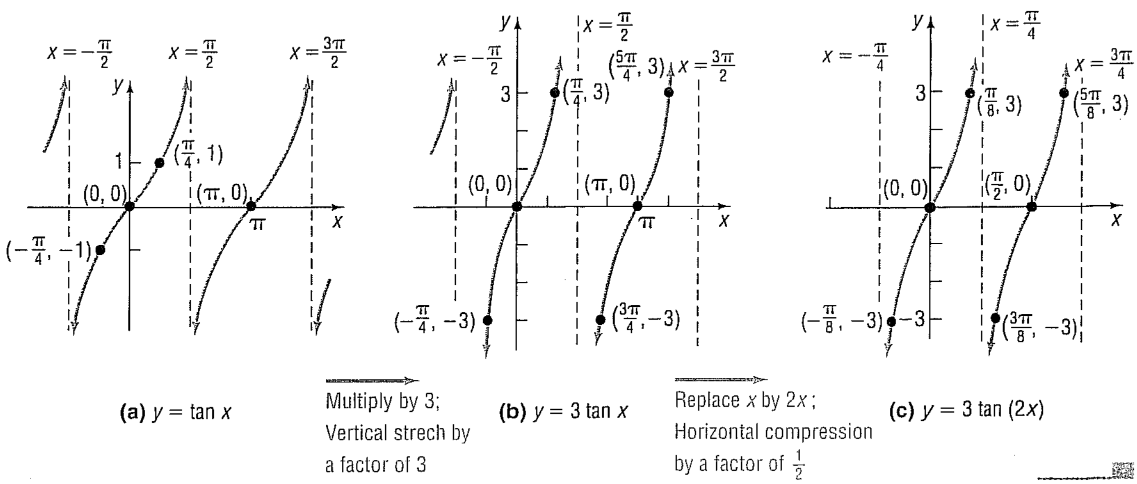


EXAMPLE 2 Graphing Functions of the Form $y = A\tan(\omega x) + B$

Graph: $y = 3\tan(2x)$

Solution Figure 65 shows the steps using transformations.

Figure 65



Notice in Figure 65(c) that the period of $y = 3\tan(2x)$ is $\frac{\pi}{2}$ due to the compression of the original period π by a factor of $\frac{1}{2}$. Also notice that the asymptotes are $x = -\frac{\pi}{4}, x = \frac{\pi}{4}, x = \frac{3\pi}{4}$, and so on, also due to the compression.

Now Work PROBLEM 21

The Graph of the Cotangent Function

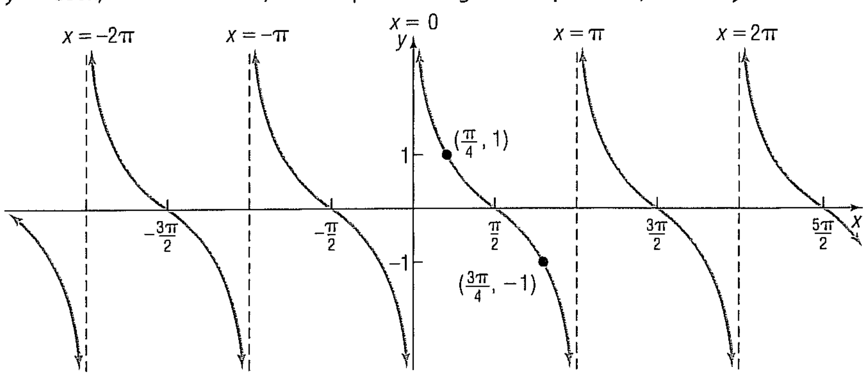
We obtain the graph of $y = \cot x$ as we did the graph of $y = \tan x$. The period of $y = \cot x$ is π. Because the cotangent function is not defined for integer multiples of π, we will concentrate on the interval $(0, \pi)$. Table 10 lists some points on the graph of $y = \cot x$, $0 < x < \pi$. As x approaches 0, but remains greater than 0, the value of $\cos x$ will be close to 1 and the value of $\sin x$ will be positive and close to 0. Hence, the ratio $\frac{\cos x}{\sin x} = \cot x$ will be positive and large; so as x approaches 0, with $x > 0$, $\cot x$ approaches ∞ $\left(\lim_{x\to 0^+} \cot x = \infty\right)$. Similarly, as x approaches π, but remains less than π, the value of $\cos x$ will be close to -1, and the value of $\sin x$ will be positive and close to 0. So the ratio $\frac{\cos x}{\sin x} = \cot x$ will be negative and will approach $-\infty$ as x approaches π $\left(\lim_{x\to \pi^-} \cot x = -\infty\right)$. Figure 66 shows the graph.

Table 10

x	$y = \cot x$	(x, y)
$\frac{\pi}{6}$	$\sqrt{3}$	$\left(\frac{\pi}{6}, \sqrt{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3}$	$\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3}$	$\left(\frac{2\pi}{3}, -\frac{\sqrt{3}}{3}\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$\frac{5\pi}{6}$	$-\sqrt{3}$	$\left(\frac{5\pi}{6}, -\sqrt{3}\right)$

Figure 66

$y = \cot x$, $-\infty < x < \infty$, x not equal to integral multiples of π, $-\infty < y < \infty$



Check: Graph $Y_1 = \cot x$ and compare the result with Figure 66. Use TRACE to see what happens when x is close to 0.

The graph of $y = A \cot(\omega x) + B$ has similar characteristics to those of the tangent function. The cotangent function $y = A \cot(\omega x) + B$ has period $\frac{\pi}{\omega}$. The role of A is to provide the magnitude of the vertical stretch; the presence of B indicates that a vertical shift is required.

Now Work PROBLEM 23

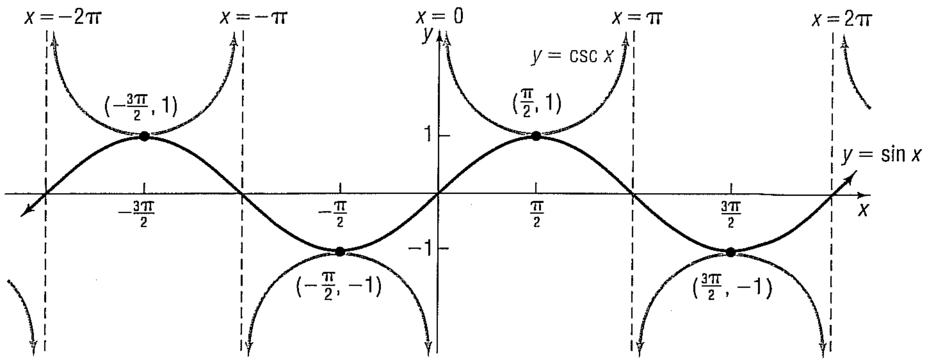
The Graph of the Cosecant Function and the Secant Function

The cosecant and secant functions, sometimes referred to as **reciprocal functions,** are graphed by making use of the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

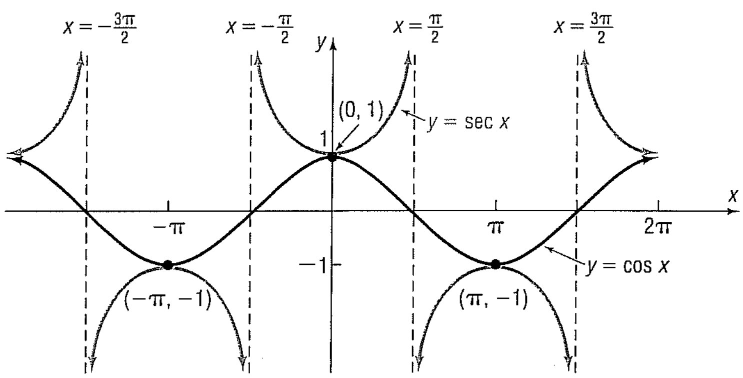
For example, the value of the cosecant function $y = \csc x$ at a given number x equals the reciprocal of the corresponding value of the sine function, provided that the value of the sine function is not 0. If the value of $\sin x$ is 0, then x is an integer multiple of π. At such numbers, the cosecant function is not defined. In fact, the graph of the cosecant function has vertical asymptotes at integer multiples of π. Figure 67 shows the graph.

Figure 67
$y = \csc x$, $-\infty < x < \infty$, x not equal to integer multiples of π, $|y| \geq 1$



Using the idea of reciprocals, we can similarly obtain the graph of $y = \sec x$. See Figure 68.

Figure 68
$y = \sec x$, $-\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $|y| \geq 1$



2 Graph Functions of the Form $y = A\csc(\omega x) + B$ and $y = A\sec(\omega x) + B$

The role of A in these functions is to set the range. The range of $y = \csc x$ is $\{y \mid |y| \geq 1\}$; the range of $y = A\csc x$ is $\{y \mid |y| \geq |A|\}$, due to the vertical stretch of the graph by a factor of $|A|$. Just as with the sine and cosine functions, the period of $y = \csc(\omega x)$ and $y = \sec(\omega x)$ becomes $\frac{2\pi}{\omega}$, due to the horizontal compression of the graph by a factor of $\frac{1}{\omega}$. The presence of B indicates a vertical shift is required.

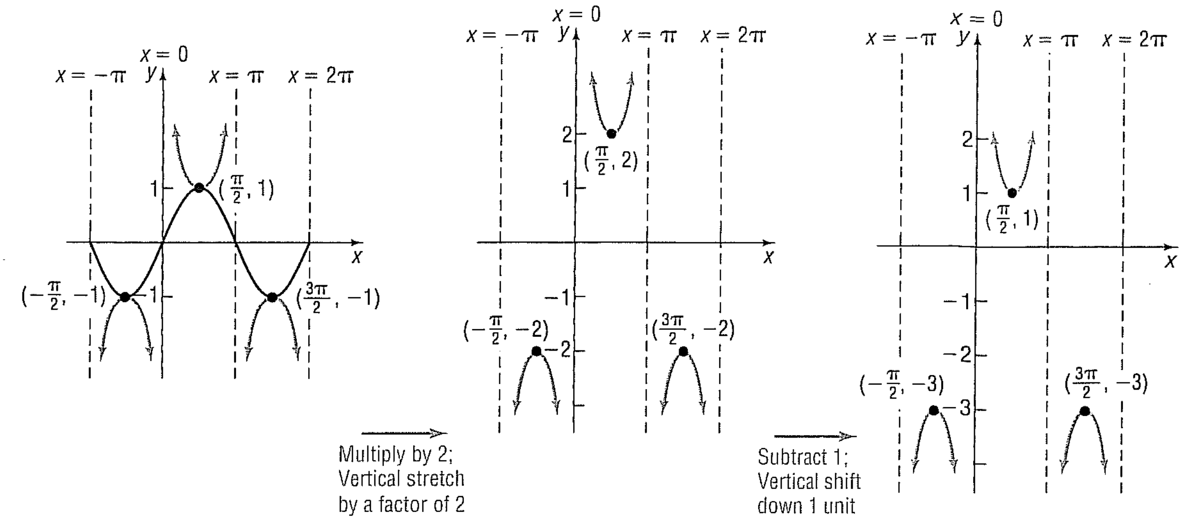
We shall graph these functions in two ways: using transformations and using the reciprocal function.

EXAMPLE 3 Graphing Functions of the Form $y = A\csc(\omega x) + B$

Graph: $y = 2\csc x - 1$

Solution Using Transformations Figure 69 shows the required steps.

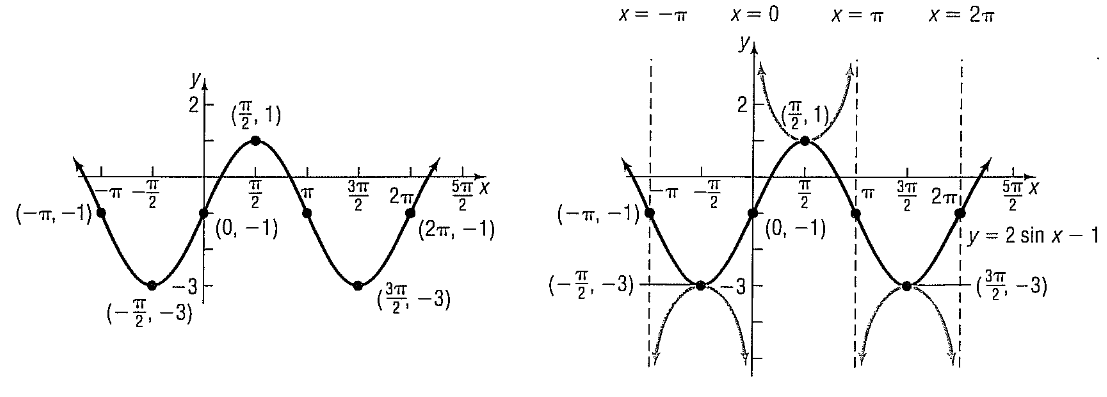
Figure 69



(a) $y = \csc x$ (b) $y = 2\csc x$ (c) $y = 2\csc x - 1$

Solution Using the Reciprocal Function We graph $y = 2\csc x - 1$ by first graphing the reciprocal function $y = 2\sin x - 1$ and then filling in the graph of $y = 2\csc x - 1$, using the idea of reciprocals. See Figure 70.

Figure 70



(a) $y = 2\sin x - 1$ (b) $y = 2\csc x - 1$

Now Work PROBLEM 29

2.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The graph of $y = \dfrac{3x - 6}{x - 4}$ has a vertical asymptote. What is it? (pp. 66–67)

2. ***True or False*** A function f has at most one vertical asymptote. (pp. 66–67)

Concepts and Vocabulary

3. The graph of $y = \tan x$ is symmetric with respect to the _____ and has vertical asymptotes at _____.

4. The graph of $y = \sec x$ is symmetric with respect to the _____ and has vertical asymptotes at _____.

5. It is easiest to graph $y = \sec x$ by first sketching the graph of _____.

6. ***True or False*** The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ each have infinitely many vertical asymptotes.

Skill Building

In Problems 7–16, if necessary, refer to the graphs to answer each question.

7. What is the y-intercept of $y = \tan x$?

8. What is the y-intercept of $y = \cot x$?

9. What is the y-intercept of $y = \sec x$?

10. What is the y-intercept of $y = \csc x$?

11. For what numbers x, $-2\pi \le x \le 2\pi$, does $\sec x = 1$? For what numbers x does $\sec x = -1$?

12. For what numbers x, $-2\pi \le x \le 2\pi$, does $\csc x = 1$? For what numbers x does $\csc x = -1$?

13. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \sec x$ have vertical asymptotes?

14. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \csc x$ have vertical asymptotes?

15. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \tan x$ have vertical asymptotes?

16. For what numbers x, $-2\pi \le x \le 2\pi$, does the graph of $y = \cot x$ have vertical asymptotes?

In Problems 17–40, graph each function. Be sure to label key points and show at least two cycles.

17. $y = 3\tan x$
18. $y = -2\tan x$
19. $y = 4\cot x$
20. $y = -3\cot x$
21. $y = \tan\left(\dfrac{\pi}{2}x\right)$
22. $y = \tan\left(\dfrac{1}{2}x\right)$
23. $y = \cot\left(\dfrac{1}{4}x\right)$
24. $y = \cot\left(\dfrac{\pi}{4}x\right)$
25. $y = 2\sec x$
26. $y = \dfrac{1}{2}\csc x$
27. $y = -3\csc x$
28. $y = -4\sec x$
29. $y = 4\sec\left(\dfrac{1}{2}x\right)$
30. $y = \dfrac{1}{2}\csc(2x)$
31. $y = -2\csc(\pi x)$
32. $y = -3\sec\left(\dfrac{\pi}{2}x\right)$
33. $y = \tan\left(\dfrac{1}{4}x\right) + 1$
34. $y = 2\cot x - 1$
35. $y = \sec\left(\dfrac{2\pi}{3}x\right) + 2$
36. $y = \csc\left(\dfrac{3\pi}{2}x\right)$
37. $y = \dfrac{1}{2}\tan\left(\dfrac{1}{4}x\right) - 2$
38. $y = 3\cot\left(\dfrac{1}{2}x\right) - 2$
39. $y = 2\csc\left(\dfrac{1}{3}x\right) - 1$
40. $y = 3\sec\left(\dfrac{1}{4}x\right) + 1$

Applications and Extensions

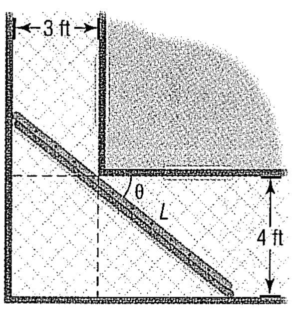
In Problems 41–44, find the average rate of change of f from 0 to $\dfrac{\pi}{6}$.

41. $f(x) = \tan x$
42. $f(x) = \sec x$
43. $f(x) = \tan(2x)$
44. $f(x) = \sec(2x)$

In Problems 45–48, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and graph each of these functions.

45. $f(x) = \tan x$; $g(x) = 4x$
46. $f(x) = 2\sec x$; $g(x) = \dfrac{1}{2}x$
47. $f(x) = -2x$; $g(x) = \cot x$
48. $f(x) = \dfrac{1}{2}x$; $g(x) = 2\csc x$

49. **Carrying a Ladder around a Corner** Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration.



(a) Show that the length L of the line segment shown as a function of the angle θ is

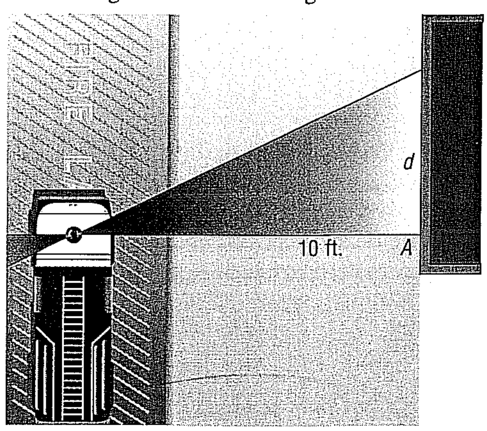
$$L(\theta) = 3\sec\theta + 4\csc\theta$$

(b) Graph $L = L(\theta)$, $0 < \theta < \dfrac{\pi}{2}$.

(c) For what value of θ is L the least?

(d) What is the length of the longest ladder that can be carried around the corner? Why is this also the least value of L?

50. **A Rotating Beacon** Suppose that a fire truck is parked in front of a building as shown in the figure.



The beacon light on top of the fire truck is located 10 feet from the wall and has a light on each side. If the beacon light rotates 1 revolution every 2 seconds, then a model for determining the distance d that the beacon of light is from point A on the wall after t seconds is given by

$$d(t) = |10\tan(\pi t)|$$

(a) Graph $d(t) = |10\tan(\pi t)|$ for $0 \le t \le 2$.

(b) For what values of t is the function undefined? Explain what this means in terms of the beam of light on the wall.

(c) Fill in the following table.

t	0	0.1	0.2	0.3	0.4
$d(t) = 10\tan(\pi t)$					

(d) Compute $\dfrac{d(0.1) - d(0)}{0.1 - 0}$, $\dfrac{d(0.2) - d(0.1)}{0.2 - 0.1}$, and so on, for each consecutive value of t. These are called **first differences.**

(e) Interpret the first differences found in part (d). What is happening to the speed of the beam of light as d increases?

51. Exploration Graph

$$y = \tan x \quad \text{and} \quad y = -\cot\left(x + \frac{\pi}{2}\right)$$

Do you think that $\tan x = -\cot\left(x + \dfrac{\pi}{2}\right)$?

'Are You Prepared?' Answers

1. $x = 4$ **2.** False

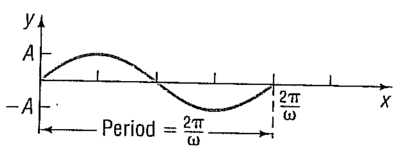
2.6 Phase Shift; Sinusoidal Curve Fitting

OBJECTIVES 1 Graph Sinusoidal Functions of the Form $y = A\sin(\omega x - \phi) + B$ (p. 158)

2 Find a Sinusoidal Function from Data (p. 162)

1 Graph Sinusoidal Functions of the Form $y = A\sin(\omega x - \phi) + B$

Figure 71
One cycle $y = A\sin(\omega x)$, $A > 0$, $\omega > 0$



We have seen that the graph of $y = A\sin(\omega x)$, $\omega > 0$, has amplitude $|A|$ and period $T = \dfrac{2\pi}{\omega}$. One cycle can be drawn as x varies from 0 to $\dfrac{2\pi}{\omega}$ or, equivalently, as ωx varies from 0 to 2π. See Figure 71.

We now want to discuss the graph of

$$y = A\sin(\omega x - \phi)$$

which may also be written as

$$y = A\sin\left[\omega\left(x - \frac{\phi}{\omega}\right)\right]$$

where $\omega > 0$ and ϕ (the Greek letter phi) are real numbers. The graph will be a sine curve with amplitude $|A|$. As $\omega x - \phi$ varies from 0 to 2π, one period will be traced out. This period will begin when

$$\omega x - \phi = 0 \quad \text{or} \quad x = \frac{\phi}{\omega}$$

and will end when

$$\omega x - \phi = 2\pi \quad \text{or} \quad x = \frac{\phi}{\omega} + \frac{2\pi}{\omega}$$

See Figure 72.

NOTE We can also find the beginning and end of the period by solving the inequality

$$0 \le \omega x - \phi \le 2\pi$$
$$\phi \le \omega x \le 2\pi + \phi$$
$$\frac{\phi}{\omega} \le x \le \frac{2\pi}{\omega} + \frac{\phi}{\omega}$$

Figure 72
One cycle $y = A\sin(\omega x - \phi)$, $A > 0$, $\omega > 0$, $\phi > 0$