Indeterminate Forms and l'Hopital's Rule

Indeterminate Forms

Limits which are undefined at the point the limit is being evaluated

Indeterminate Quotients: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Indeterminate Products: $0(\pm\infty)$

Indeterminate Differences: $\infty - \infty$

Indeterminate Powers: 0^0 or ∞^0 or 1^∞

Indeterminate Quotients: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Question often arises "Why isn't the latter 1 or –1?"

Following examples may help:

$$\lim_{x\to\infty} \frac{x-1}{x^2-1} = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} \frac{x^2-1}{2x^2-1} = \frac{\infty}{\infty}$$

We know how to evaluate the limit by multiplying the denominator and numerator of each by $1/x^2$.

$$\lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{0}{1} = 0 \qquad \lim_{x\to\infty} \frac{1 - \frac{1}{x^2}}{2 - \frac{1}{x^2}} = \frac{1}{2}$$

Algebraic methods are not always applicable.

$$\lim_{x \to 0} \frac{\sin x}{x} \quad \text{or} \quad \lim_{x \to \infty} \frac{\ln x}{x}$$

L'Hopital's Rule:

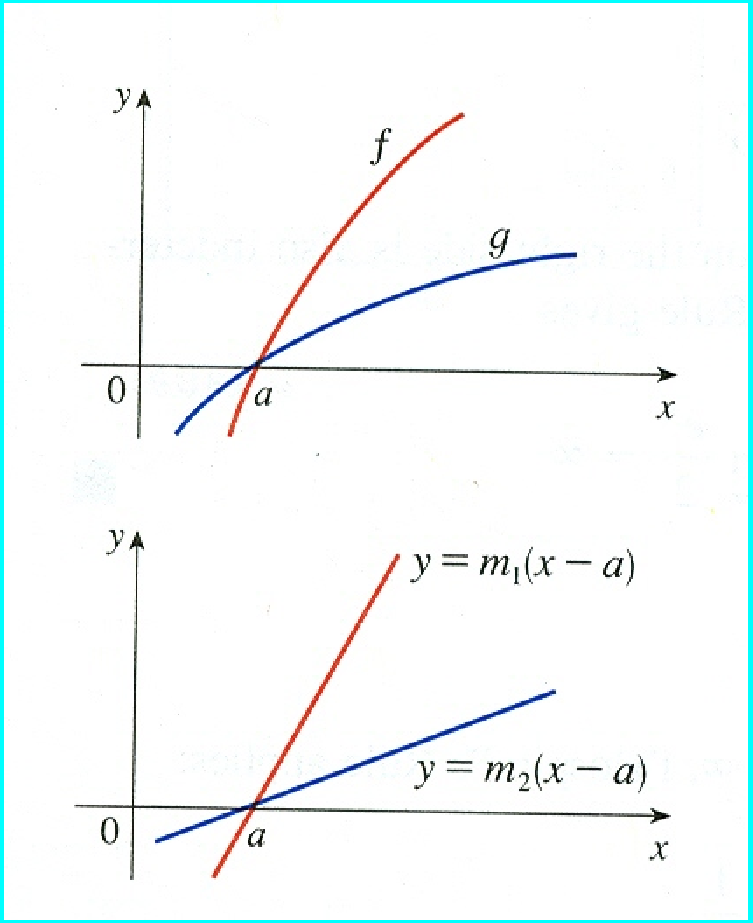
If f and g are differentiable and g'(x) ≠0 near x, and:

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ **If it exists.**



Close to a f and g are aprox. linear

If f(a) = g(a) = 0

$$\frac{f(x)}{g(x)} = \frac{m_f\,(x-a)}{m_g\,(x-a)}$$

$$= \frac{m_f\,\cancel{(x-a)}}{m_g\,\cancel{(x-a)}}$$

$$= \frac{f'(x)}{g'(x)}$$

$$\text{Ex: } \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

$$\text{Ex: } \lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\left(\frac{1}{x}\right)}{1} = \lim_{x\to 1}\frac{1}{1} = 1$$

$$\text{Ex: } \lim_{x\to \infty}\frac{e^x}{x^2} = \lim_{x\to \infty}\frac{e^x}{2x} = \lim_{x\to \infty}\frac{e^x}{2} = \infty$$

$$\text{Ex: } \lim_{x\to \infty}\frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to \infty}\frac{\left(\frac{1}{x}\right)}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to \infty}\frac{3}{\sqrt[3]{x}} = 0$$

Ex: $$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

$$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$$

$$= \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Ex: $$\lim_{x\to \pi}\frac{\sin x}{1-\cos x} \neq -\infty$$

$$= \frac{0}{2} = 0$$

Indeterminate Products: $0 \times \infty$

Unclear what the product will be.
Depends on rate of approach to each value.
<u>Remember:</u> It is the product as a value is approached not the product at the point.

Handle by rewriting as a quotient and apply l'Hopital's Rule

$$fg = \frac{f}{\left(\frac{1}{g}\right)} \text{ or } \frac{g}{\left(\frac{1}{f}\right)}$$

Ex: $$\operatorname*{Limit}_{x \to 0^+} (x \ln x) = \operatorname*{Limit}_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \operatorname*{Limit}_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \operatorname*{Limit}_{x \to 0^+} (-x) = 0$$

Indeterminate Differences: $\infty - \infty$

Convert the difference to a quotient.

Ex: $\text{Limit}_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \text{Limit}_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$

$$= \text{Limit}_{x \to \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x} \right)$$

$$= \text{Limit}_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin x} = 0$$

Indeterminate Powers: 0^0 or ∞^0 or 1^∞

Let $y = [f(x)]^{g(x)}$ and take log of each side.

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} \qquad y = (1+\sin 4x)^{\cot x}$$

$$\ln y = \cot x \ln(1+\sin 4x) = \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{\left(\dfrac{4\cos 4x}{1+\sin 4x}\right)}{\sec^2 x} = 4$$

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

Ex:

$$\lim_{x\to 0^+}\left(x^x\right) \qquad y = x^x$$

$$\ln y = x \ln x$$

$$\lim_{x\to 0^+}\left(\ln y\right) = \lim_{x\to 0^+}\left(x \ln x\right)$$

$$= \lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{x}\right)} = \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\left(-x\right) = 0$$

$$\lim_{x\to 0^+}\left(y\right) = \lim_{x\to 0^+} e^{\ln y} \Rightarrow e^0 \Rightarrow 1$$